DOMINANT BALANCE ANALYSIS OF THE FRACTIONAL POWER
DAMPED HARMONIC OSCILLATOR

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Abstract. We apply the principle of dominant balance to determine the solution, for which there is finite time dynamics, in the neighborhood of the extinction of the oscillations, for a simple harmonic oscillator with fractional power damping. This work extends the previous calculations of Mickens [2,5].

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1. Introduction

The simple harmonic oscillator (SHO) with cube root damping

$$\ddot{x} + x = -\epsilon (\dot{x})^{\frac{1}{3}}, \quad \epsilon > 0 \quad (1.1)$$

has been investigated by Mickens and his collaborators [2,5], with a generalization to Duffing type nonlinear oscillators [4]. The general dynamics of Eq.(1.1) has been discussed in Mickens [2]. In summary,

(i) In the x-y phase-plane, the system of differential equations are

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x - \epsilon y^{\frac{1}{3}}. \quad (1.2)$$

(ii) There is a single fixed-point

$$(\bar{x}, \bar{y}) = (0, 0), \quad (1.3)$$

and it is (globally) stable.

(iii) In general, the solutions oscillate with decreasing amplitude to the fixed-point.

(iv) For $0 < \epsilon \ll 1$, the application of perturbation theory, i.e., the method of averaging, shows that the system has finite-time dynamics. This means that there exists a time, t^*

$$t^* = f(\epsilon, x_0), \quad (1.4)$$

with initial conditions

$$x_0 = x(0), \quad \dot{x}(0) = y(0) = 0, \quad (1.5)$$

such that

$$x(t) = 0, \quad t \geq t^*. \quad (1.6)$$

(v) There are two time scales associated with this system. The natural periodic oscillation of the SHO, i.e., Eq.(1.1) with $\epsilon = 0$, and the time scale of the damping. These are, respectively,

$$T_{SHO} = 2\pi, \quad T_0 = \left(\frac{x_0^{\frac{2}{3}}}{\epsilon} \right)^{\frac{3}{5}}, \quad (1.7)$$

where

$$T_{SHO} \ll T_0, \quad 0 < \epsilon \ll 1. \quad (1.8)$$

This short note has two goals: First, generalize the damping to any fractional power of the damping term, i.e., Eq.(1.1) becomes

$$\ddot{x} + x = -\epsilon (\dot{x})^p, \quad \epsilon > 0, \quad (1.9)$$

and

$$0 < p < 1. \quad (1.10)$$

Second, calculate the analytic form of the solution to Eq.(1.9) in the neighborhood of its extinction time t_p^* . (Note that the mathematical form of the dominant balance ansatz incorporates the assumptions of a finite time extinction.)

In outline, Section 2 provides preliminary information and references to time scales and the dominant balance method, which is then used in Section 3 to evaluate the required(asymptotic) solution. We summarize our conclusions in Section 4.

2. Preliminaries:

To eliminate the need for absolute value evaluations and other related issues, we select p to be

$$p = \frac{2m+1}{2n+1}; \quad (m,n) : 0, 1, 2, 3, \dots, \text{ with } 0 \leq m < n, \quad (2.1)$$

and this gives the restriction of Eq. (1.10). We call these fractions "odd", in the sense that x^p is an odd function, i.e., it can be shown that

$$(-x)^p = -x^p. \quad (2.2)$$

Note that $q = 1 - p$ is an "even" fraction.

The method of dominant balance is a technique for formulating and calculating the asymptotic behavior of differential and many other many other types of equations. See Mickens [3] for a summary of this methodology along with references to its applications.

3. Dominant Balance Analysis:

The use of the averaging method on

$$\ddot{x} + x = - \in (x)^p, \quad 0 < p < 1, \quad (3.1)$$

shows that the solutions have properties similar to the particular case $p = \frac{1}{3}$, i.e., the solutions become zero in a finite time. Thus, we take the dominant balance ansatz to be

$$x(t) = \begin{cases} A(t^* - t)^\beta, & t^* - \delta < t \leq t^* \\ 0, & t > t^*, \end{cases} \quad (3.2)$$

for some $\delta > 0$ and for constants (A, β) to be determined. Note that t^* can not be determined by this procedure since the differential equation is invariant under $t \rightarrow t + t_0$, where t_0 is arbitrary.

Substituting Eq.(3.2) into Eq. (3.1) gives

$$\beta(\beta - 1)A(t^* - t)^{(\beta-2)} + A(t^* - t)^\beta \sim - \in \beta^p(t^* - t)^{(\beta-1)p}. \quad (3.3)$$

For the balancing there are three possibilities:

(I)

$$\beta(\beta - 1)A(t^* - 1)^{(\beta-2)} + A(t^* - t)^\beta \sim 0, \quad (3.4)$$

$$(II) \quad A(t^* - t)^\beta \sim \in \beta^p A^p (t^* - t)^{(\beta-1)p}, \quad (3.5)$$

$$(III) \quad \beta(\beta - 1)A(t^* - t)^{(\beta-2)} \sim \in \beta^p A^p (t^* - t)^{(\beta-1)p}. \quad (3.6)$$

Equating the coefficients and powers gives

$$(I) \quad \beta - 2 = \beta \quad \text{or no solution possible;} \\ (II) \quad \beta = (\beta - 1)p \quad \text{or} \quad \beta = -\left(\frac{p}{1-p}\right) < 0. \quad (3.7)$$

For this case, there is a singularity at $t = t^*$, consequently, the solution is unbounded. Thus, this case can be ruled out.

(III) $\beta - 2 = (\beta - 1)p$ and $\beta(\beta - 1)A = \beta^p A^p$
and, we calculate β and A to be

$$\beta = \frac{2-p}{1-p} > 1, \quad A = \left(\frac{1}{\beta}\right) \left(\frac{\in}{\beta-1}\right) \left(\frac{1}{1-p}\right). \quad (3.8)$$

Next, we need to check that the following two conditions hold

$$\beta > \beta - 2, \quad \beta > (\beta - 1)p, \quad (3.9)$$

i.e., the neglected term must be smaller as $t \rightarrow t^*$, from below t^* . A simple calculation shows both inequalities hold.

Putting these results together gives the following representation of the solution, $x(t)$, in a neighborhood of t^* :

$$x(t) \sim \begin{cases} A(t^* - t)^\beta, & t^* - \delta < t \leq t^*, \delta > 0, \\ 0, & t > t^*, \end{cases} \quad (3.10)$$

where (A, β) are given in the expressions of Eq. (3.8).

4. Discussion

Combining the results of Mickens[2,5], we conclude that the SHO with non-linear, frictional power damping has finite-time dynamics. Further, we have determined a mathematical expression for the solution near the extinction time t^* ; see Eq.(3.10). Thus, the motion of this oscillator is decreasing and oscillatory [2,5].

Finally, it should be indicated that our results are not in agreement with those obtained by Amann and Diaz [1]. We have examined their paper [1], but can not follow their arguments.

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