LECTURES IN THE THEORY OF FUNCTION OF A COMPLEX VARIABLE PART III

A THESIS

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BY

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PREFACE

Recently certain students in the Department of Mathematics of Atlanta University became intensely interested in The Theory of Functions of a Complex Variable. As a result of this interest a series of theses was initiated. Each thesis attempts to simplify and clarify a particular portion of the lecture notes obtained while enrolled in the course.

This thesis, the third in the series, presents the treatment of the first phase of the second semester course in The Theory of Functions of a Complex Variable. It is a continuation of <u>Lectures in The Theory of Functions of a Complex Variable</u>, Part I and Part II, theses by Lindsey Branch Johnson and David Lee Hunter.

This paper deals extensively with the calculus of the residues, and conformal representation, making reference to certain related theorems where necessary.

It is the sincere hope of the writer that this paper will be helpful and serve as an inspiration to those interested in The Theory of Functions of a Complex Variable.

J. L. S., JR.

TABLE OF CONTENTS

LIST OF SYMBOLS

Listed below are the symbols used in this paper and a statement of their meaning.

$\longrightarrow$ approaches or goes over to

$\Longrightarrow$ implies

ϵ epsilon

$\neq$ not equal to

$=$ equal to

$\stackrel{d}{=}$ defined to be

$<$ less than

$\leq$ less than or equal to

$>$ greater than

$\geq$ greater than or equal to

CHAPTER I

THE CALCULUS OF RESIDUES

The Residue Theorem.- Suppose f is regular in a neighborhood U of a point Z_0, then by Cauchy's integral theorem, if C is a closed path contained in U,

$$\int_C f(z)\,dz = 0$$

Let f be singled valued and regular in a neighborhood U about Z, except possibly at the point Z_0 itself, and let C be a closed path contained in U such that Z_0 is contained in C, then

$$\int_C f(z)\,dz$$

is not necessarily equal to zero. Its value can readily be determined however, since f(Z) can be expanded in a Laurent series in a neighborhood of Z_0($0<|Z - Z_0|<r$), that is to say

$$f(Z)=\sum_{n=-\infty}^{+\infty} a_n\,(Z-Z_0)^n=\cdots+a_{-2}(Z-Z_0)^{-2}+a_{-1}(Z-Z_0)^{-1}+a_0+a_1(Z-Z_0)+\cdots$$

in ($0<|Z-Z_0|<r$), and we have

$$\int_{C:|z-z_0|=\rho} f(z)\,dz = 2\pi i\, a_{-1}\,,\quad \text{where } 0<\rho<r.$$

This is true since the integral of each term of the expansion except $a_{-1}(Z-Z_0)^{-1}$ is zero. So that we have

$$\int_c \frac{a_{-1}}{z-z_0}\, dz = a_{-1}\left[\int_c \frac{dz}{z-z_0}\right] = 2\pi i\, a_{-1}, \text{ hence}$$

$$\frac{1}{2\pi i}\int_c f(z)\, dz = a_{-1}\,.$$

We now define the residue of f(Z).

Definition.- The coefficient of that term of the Laurent expansion whose exponent is -1 is called the residue of f(Z) at the point Z_0. That is,

$$a_{-1} = \frac{1}{2\pi i}\int_c f(z)dz$$

is the residue of f at Z_0, where C is a simple, closed positively oriented path contained in the domain of regularity of f, containing the point Z_0 in its interior. Now we state and prove

Theorem I. The Residue Theorem.- Let the function f be regular and single-valued in a region G, except for a finite number of poles. Let C be a simple closed, positively oriented path contained in G, not passing through any poles of f, then

$$\frac{1}{2\pi i}\int_c f(z)\, dz = \sum R,$$

where ΣR is the sum of the residues of $f(Z)$ at its poles inside C.

Proof.- Let $Z_1, Z_2, \ldots, Z_k$ be the finite number of poles of $f(Z)$ inside C. Let C_i, $(i=1, 2, \ldots, k)$ be circles about Z_i as center, such that $C_i \cap C_j = 0$ if $i \neq j$, $(j = 1, 2, \ldots, k)$.

By an extension of Cauchy's integral theorem we have

$$\frac{1}{2\pi i}\int_C f(z)\,dz = \frac{1}{2\pi i}\left[\int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots + \int_{C_k} f(z)\,dz\right].$$

Therefore our theorem now follows, since the residues in question are the terms of the right member of this equation.

Q. E. D.

This theorem has numerous applications. We first consider a few chosen at random.

(a) Suppose the function f is regular and single valued in a region G, except for a finite number of poles. Let C be a simple, closed, positively oriented path contained in G, suth that $f(Z)$ 0 on C and C does not pass through any pole of $f(Z)$, then we make the following assertion :

Theorem 2.-

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N - P,$$

Where N is the number of zeros and P is the number of poles of $f(Z)$ inside C.

Proof.- Let $Z_1, Z_2, \ldots, Z_k$ be the zeros of $f(Z)$ inside

C and b_1, b_2, . . . , b_k be their respective multiplicites. Let $a_1, a_2, \ldots, a_m$ be the poles of $f(Z)$ inside C and let h_1, h_2, . . . , h_m be their respective orders. Then observe that $f(Z)$ can be written as follows:

$$\text{(i)}\quad f(Z) = (Z-Z_j)^{bj}\, P_j(Z), \quad (j = 1, 2, \ldots, k)$$

where $P_j(Z) \neq 0$ for $Z = Z_j$, and

$$\text{(ii)}\quad f(Z) = 1/(Z-a_i)^{hi} Q_i(Z), \quad (i = 1, 2, \ldots, m)$$

where $Q_i(Z)$ is regular at $Z = a_i$. Moreover from (i)

$$f'(Z)/f(Z) = b_j/Z-Z_j + P'_j(Z)/P_j(Z), \quad (j = 1, 2, \ldots, k)$$

and from (ii)

$$f'(Z)/f(Z) = -h_i/Z-a_i + Q'_i(Z)/Q_i(Z), \quad (i = 1, 2, \ldots, m).$$

Hence the function $f'(Z)/f(Z)$ has inside C, simple poles at $Z_1, Z_2, \cdots, Z_k, a_1, a_2, \cdots, a_m$. Therefore by the residue theorem

$$\frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)} = \sum_{j=1}^{k} b_j - \sum_{i=1}^{m} h_i = N - P,$$

where $N = \sum b_j$ and $P = \sum h_i$. Q.E.D.

(b) Evaluate certain integrals by the residue theorem. Show that

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \pi.$$

Note: Before we proceed we make the following remarks:

(i) If $f(Z)$ has a simple pole at Z_0, then its Laurent expansion in the neighborhood $(0 < |Z-Z_0| < r)$ is as follows:

$$f(Z) = a_{-1}/(Z-Z_0) + a_0 + a_1(Z-Z_0) + \cdots, \text{ and}$$

$$(Z-Z_0)f(Z) = a_{-1} + a_0(Z-Z_0) + a_1(Z-Z_0)^2 + \cdots.$$

Thus we have in this case

$$a_{-1} = \lim_{Z \to Z_0} \left[(Z-Z_0)f(Z)\right].$$

(ii) If $f(Z)$ has a pole of order b, $(1 < b < +\infty)$ at Z_0, then its Laurent expansion in the neighborhood $(0 < |Z-Z_0| < r)$ is as follows:

$$f(Z) = a_{-b}/(Z-Z_0)^b + a_{-b+1}/(Z-Z_0)^{b-1} + \cdots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(Z-Z_0) + \cdots,$$

and

$$f(Z)(Z-Z_0)^b f(z) = a_{-b} + a_{-b+1}(Z-Z_0) + \cdots + a_{-1}(Z-Z_0)^{b-1} + \cdots.$$

Thus we have in this case

$$a_{-1} = 1/(b-1)! \left\{\lim d^{b-1}/dZ^{b-1}\left[(Z-Z_0)^b f(Z)\right]\right\}.$$

Now we evaluate

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}.$$

Let Γ_r be the closed path which contains the segment from $-R$ to $+R$ and the upper semicircle C_R, $|Z| = R$ back to $-R$, (with the orientation indicated in figure 1).

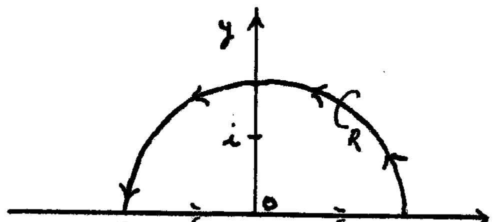

FIG. 1

By the residue theorem

$$\int_{\Gamma_R} \frac{dz}{1+z^2} = 2\pi i \sum R\left(\frac{1}{1+z^2}\right),$$

where $\sum R(1/(1+Z^2))$ denotes the sum of the residues of $1/(1+Z^2)$ inside Γ_R. Since $1/(1+Z^2) = 1/(Z+i)(Z-i)$, we see that the poles of $1/(1+Z^2)$ are $\pm i$. Only i is in the upper half-plane, so we choose R large enough so that it contains i. Now upon applying note (i) from page 5 we have

$$\mathrm{Res}_i \left[\frac{1}{1+z^2}\right]^1 = \lim_{z\to i} \left[(z-i)\frac{1}{1+z^2}\right] = \frac{1}{2i}.$$

Where ($\mathrm{Res}\ i$) denotes the residue of $1/(1+Z^2)$ at i. Note that in this case $f(Z) = 1/(1+Z^2)$. Hence

$$\int_{\Gamma_R} \frac{dz}{1+z^2} = 2\pi i\left(\frac{1}{2i}\right) = \pi.$$

But

$$\int_{\Gamma_R} \frac{dz}{1+z^2} = \int_{C_R} \frac{dz}{1+z^2} + \int_{-R}^{+R} \frac{dx}{1+x^2}.$$

We want to show that

$$\int_{C_R} \frac{dz}{1+z^2} \longrightarrow 0, \text{ as } R \longrightarrow +\infty.$$

Let R be fixed but greater than one. Then observe that

$$\left|\int_{C_R} \frac{dz}{1+z^2}\right| \leq \frac{1}{R^2-1}\pi R, \text{ since } \left|\frac{1}{1+z^2}\right| \leq \frac{1}{R^2-1}$$

on C_R. Therefore

$$\int_{C_R} \frac{dz}{1+z^2} \longrightarrow 0, \text{ as } R \longrightarrow +\infty.$$

Consequently $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \pi.$

We now turn our attention to more general applications of the residue theorem for the evaluation of certain integrals.

1. Evaluation of a type of Infinite Integral

Theorem 3.- Let $Q(Z)$ be a function of Z satisfying the following conditions:

(i) $Q(Z)$ is meromorphic[1] in the upper half-plane;

(ii) $Q(Z)$ has no poles on the real axis;

(iii) $ZQ(Z) \longrightarrow 0$ uniformly as $|Z| \longrightarrow +\infty$, for $0 \leq \arg Z \leq \pi$;

(iv) $$\int_{-\infty}^{0} Q(x)\,dx \quad \text{and} \quad \int_{0}^{+\infty} Q(x)\,dx$$

both converge.

Then

$$\int_{-\infty}^{+\infty} Q(x)\,dx = 2\pi i \sum R^+,$$

[1]Only singularities in finite part of plane are poles.

where $\sum R^+$ denotes the sum of the residues of $Q(Z)$ at its poles in the upper half-plane.

Proof: Consider the semicircle C_R: $|Z|=R$, $0 \leq \arg Z \leq \pi$. Then consider the closed curve $\Gamma_R = C_R + (-R,R)$. (See figure 2).

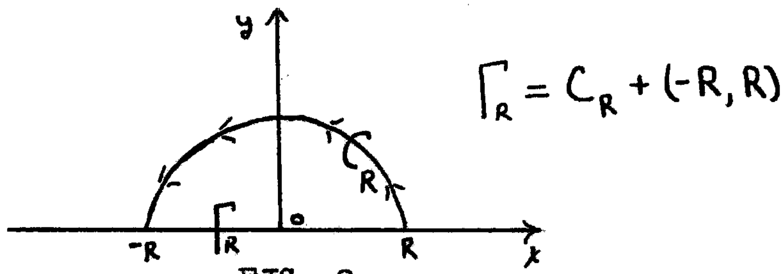

FIG. 2

Let R be large enough so that Γ_R contains in its interior all the poles of $Q(Z)$. Then, by the residue theorem,

$$\int_{\Gamma_R} Q(z)\,dz = 2\pi i \sum R^+.$$

But

$$\int_{\Gamma_R} Q(z)\,dz = \int_{C_R} Q(z)\,dz + \int_{-R}^{0} Q(x)\,dx + \int_{0}^{R} Q(x)\,dx).$$

we want to show that

$$\int_{C_R} Q(z)\,dz \longrightarrow 0, \text{ as } R \longrightarrow \infty.$$

Observe that

$$\left|\int_{C_R} Q(z)\,dz\right| \leq \max_{\text{on } C_R} |Q(z)|\,\pi R.$$

By condition (iii) of our theorem, there exist an R_o such that

$|Z|\cdot|Q(Z)| < \epsilon$ for $|Z| > R_0$. Hence for all $|R| > R_0$ we have

$$\left| \int_{C_R} Q(z)\, dz \right| < \pi \epsilon .$$

Thus

$$\int_{C_R} Q(z)\,dz \longrightarrow 0 \text{ as } R \longrightarrow \infty .$$

If (iv) is satisfied, it follows that

$$\int_{-\infty}^{+\infty} Q(x)\, dx = 2\pi i \sum R^{+}.$$

Q.E.D.

Remarks.- Suppose $Q(Z)$ $N(Z)/D(Z)$, where $N(Z)$ and $D(Z)$ are polynomials and the degree of the polynomial $D(Z)$ exceeds that of $N(Z)$ by at least two and $D(Z)$ is not zero when imaginary Z is zero. Then conditions (i), (ii), (iii) and (iv) of the previous theorem are satisfied.

Example.- Evaluate

$$\int_0^{\infty} \frac{dx}{x^4 + a^4} \quad , \text{ where } a > 0.$$

Set $Q(Z) = 1/(Z^4) + a^4$. The poles of $Q(Z)$ will be the zeros of $Z^4 + a^4$, that is to say, the solutions of $Z^4 + a^4 = 0$. We find these solutions as follows:

$Z^4 = -a^4$, (where $e^{\pi i} = \cos\pi + i \sin\pi = -1$), and

$Z^4 = e^{\pi i} a^4$.

Thus

$Z^4 = a^4 e^{(2\pi k + \pi) i} = a^4 e^{(2k+1)\pi i}$,(where $k = 0, \pm 1, \pm 2, \ldots$). Hence $Z_k = a\, e^{\left(\frac{2k+1}{4}\right)\cdot \pi i}$, (where $k = 0, 1, 2, 3,$).

Now upon substituting for k we have

$$Z_0 = a e^{\frac{\pi i}{4}},\ Z_1 = a e^{\frac{3}{4}\pi i},\ Z_2 = a e^{\frac{5}{4}\pi i} \text{ and } Z_3 = a e^{\frac{7}{4}\pi i},$$

where only Z_0 and Z_1 are in the upper half-plane.

Observe

$$Z_0 = a e^{\frac{\pi i}{4}} = a\frac{\sqrt{2}}{2}(1+i) \text{ and}$$

$$Z_1 = a e^{\frac{3}{4}\pi i} = a\frac{\sqrt{2}}{2}(-1+i)$$

Note that $Z/(Z^4 + a^4) \longrightarrow 0$ uniformly as $|Z| \longrightarrow +\infty$.

Hence the previous theorem is applicable. Therefore

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+a^4} = 2\pi i \sum R^+ ,$$

that is

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+a^4} = 2\pi i \sum (\text{residues at } Z = a e^{\pi i/4}, a e^{3/4 \pi i}).$$

But the residues at the simple poles $Z = Z_i, (i = 0, 1)$ are

$$\lim_{z \to z_i} \left[\frac{z - z_i}{z^4 - z_i^4}\right],\ i = 0,\ 1.$$

Since our limit is an expression of the indeterminate form[2] 0/0, we apply L'Hospital's rule, and so

$$\lim_{z \to z_i} \left[\frac{z - z_i}{z^4 - z_i^4}\right] = \lim_{z \to z_i} \frac{1}{4z^3} = \frac{1}{4z_i^3}.$$

Thus

$$\operatorname*{Res.}_{z_0}\left[\frac{1}{z^4+a^4}\right] = \frac{1}{4}\left[\frac{1}{a^3 e^{3/4 \pi i}}\right] = \frac{1}{4a^3} \cdot e^{-\frac{3}{4}\pi i}$$

[2] For indeterminate forms see, John M. H. Olmsted, Intermediate Analysis (Appleton-Century-Crofts, Inc., 1956).

$$= \frac{\sqrt{2}}{8a^3}(1-i)$$

and

$$\operatorname*{Res}_{z_1}\left[\frac{1}{z^4+a^4}\right] = \frac{1}{4}\left(\frac{1}{a^3 e^{\frac{1}{4}\pi i}}\right) = \frac{1}{4a^3}\cdot e^{-\frac{1}{4}\pi i} = \frac{\sqrt{2}}{8a^3}(-1-i)$$

Hence $\sum R^+ = \frac{-i\sqrt{2}}{4a^3}$, and $\int_{-\infty}^{+\infty}\frac{dx}{x^4+a^4} = \frac{\pi\sqrt{2}}{2a^3}$

Recall

$$\int_0^\infty \frac{dx}{x^4+a^4} = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{dx}{x^4+a^4}.$$

Therefore

$$\int_0^\infty \frac{dx}{x^4+a^4} = \frac{1}{2}\left(\frac{\pi\sqrt{2}}{2a^3}\right) = \frac{\pi\sqrt{2}}{4a^3}.$$

Exercise.- Use theorem 3 to prove the following results:

(1) $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{(x^2+b^2)(x^2+c^2)^2} = \frac{\pi(b+2c)}{2bc^3(b+c)^2}$, $(b>0, c>0)$.

(2) $\displaystyle\int_0^\infty \frac{x^6\,dx}{(a^4+x^4)^2} = \frac{3\sqrt{2}\pi}{16a}$, $(a>0)$.

2. Integration Round the Unit Circle

Suppose $\phi(u,v)$ is a function of u and v, where $u^2 + v^2 = 1$.

In particular consider

(i) $\displaystyle\int_0^{2\pi}\phi(\sin\theta, \cos\theta)\,d\theta,$

where $\phi(\sin\theta, \cos\theta)$ is a rational function of $\sin\theta$ and cos θ.

Set $Z = e^{i\theta}$, but $e^{i\theta} = \cos\theta + i\sin\theta$, and $e^{-i\theta} =$ $\cos\theta - i\sin\theta$. Substracting these two quantities we have

$$\frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin\theta,$$

that is to say

$$\sin\theta = 1/2i(Z - 1/Z).$$

Using a similar procedure, we obtain

$$\cos\theta = 1/2(Z + 1/Z).$$

Since $dZ = i\,e^{i\theta}d\theta$, we have that $d\theta = dZ/i\,e^{i\theta} = dZ/iZ$.

Now upon substituting in (1) the values obtained above and setting the results equal to I, we have

$$I = \int_{|z|=1} \phi\left[1/2i(Z - 1/Z),\ 1/2(Z + 1/Z)\right] dZ/iZ.$$

Hence $I = 2\pi i\left\{\text{sum of the residues of } \phi\left[1/2i(Z - 1/Z)/iZ,\ 1/2(Z + 1/Z)/iZ\right] \text{inside C}\right\}$,

where C is the unit circle $|Z| = 1$. Let $\sum R_c$ denote the sum of the residues of $\phi\left[1/2i(Z - 1/Z)/iZ,\ 1/2(Z + 1/Z)/iZ\right]$ at its poles inside C. Then

$$I = 2\pi i \sum R_c.$$

Example.- Prove that, if $a > b > 0$,

$$J \equiv \int_0^{2\pi} \frac{\sin^2\theta\, d\theta}{a + b\cos\theta} = \frac{2\pi}{b^2}\left\{a - \sqrt{(a^2 - b^2)}\right\}^2.$$

Proof: Set $Z = e^{i\theta}$. Then $\cos\theta = 1/2(Z + 1/Z)$, $\sin\theta = 1/2i(Z - 1/Z)$ and $d\theta = dZ/iZ$. Now upon making the above change of variable, if C is the unit circle $|Z| = 1$,

$$J = \frac{i}{2b} \int_C \frac{(z^2-1)^2\, dz}{z^2(z^2 + 2\frac{a}{b} z + 1)},$$

consider $Z^2 + 2\,a/(b)\,Z + 1 = 0$, then

$$Z = \frac{-\frac{2a}{b} \pm \sqrt{\frac{4a^2}{b^2} - 4}}{2} = \frac{-a \pm \sqrt{a^2 - b^2}}{b}.$$

Set

$$\alpha = \frac{-a + \sqrt{a^2 - b^2}}{b} \quad \text{and} \quad \beta = \frac{-a - \sqrt{a^2 - b^2}}{b}$$

that is to say α and β are the roots of the quadratic

$Z^2 + 2\,a/(b)\,Z + 1 = 0.$

Observe that $\alpha\beta = 1$ and $\alpha + \beta = -2\,a/(b)$. Since the product of the roots α, β is unity, we have $|\alpha|\cdot|\beta| = 1$, where $|\beta| > |\alpha|$, and so $Z = \alpha$ is the only simple pole inside C. The origin is a pole of order two. We calculate the residues at (i) $Z = \alpha$ and (ii) $Z = 0$.

(i) $$\operatorname{Res}_{\alpha}\left[\frac{(z^2-1)^2}{z^2(z^2 + 2\frac{a}{b} z + 1)}\right] = \lim_{z \to \alpha} \left\{\frac{(z-\alpha)(z^2-1)^2}{z^2(z-\alpha)(z-\beta)}\right\},$$

$$= \lim_{z \to \alpha}\left[\frac{(z^2-1)^2}{z^2(z-\beta)}\right] = \frac{(\alpha - \frac{1}{\alpha})^2}{\alpha - \beta} = \frac{(\alpha-\beta)^2}{\alpha-\beta}$$

$$= \alpha - \beta = \frac{-a + \sqrt{a^2-b^2}}{b} + \frac{a + \sqrt{a^2-b^2}}{b} = \frac{2\sqrt{a^2-b^2}}{b}$$

(ii) $\operatorname{Res}_0\left[\frac{(z^2-1)^2}{z^2(z^2 + 2\frac{a}{b} z + 1)}\right]$ is the coefficient of $1/Z$ in the Laurent expansion of $\frac{(z^2-1)^2}{z^2(z^2 + 2\frac{a}{b} z + 1)}$

in $0 < |Z| < r$, where $(r > 0)$. But

$$\frac{(z^2-1)^2}{z^2(z^2+2\frac{a}{b}z+1)} = \frac{1-2z^2+z^4}{z^2(1+2\frac{a}{b}z+z^2)}.$$

It is easily seen that the coefficient of $1/Z$ is $-2\,a/b$. Hence

$$J = \frac{i}{2b}\cdot 2\pi i \sum R_e ,$$

where

$$\sum R_e = -\frac{2a}{b} + \frac{2\sqrt{a^2-b^2}}{b}.$$

Therefore

$$J = \frac{i}{2b}\cdot 2\pi i\left\{-\frac{2a}{b} + \frac{2\sqrt{a^2-b^2}}{b}\right\} = \frac{\pi}{b^2}\left\{a - 2\sqrt{a^2-b^2}\right\}$$

Q.E.D.

Exercise.- Use the above method to prove the following results:

(i) $$\int_0^{\pi} \frac{a\,d\theta}{a^2+\sin\theta} = \frac{\pi}{\sqrt{1+a^2}}, \quad (a>0).$$

(ii) $$\int_0^{2\pi} \frac{\cos^2 3\theta\,d\theta}{1-2p\cos 2\theta+p^2} = \pi\frac{1-p+p^2}{1-p}, \quad (0<p<1).$$

(iii) $$\int_0^{2\pi} \frac{(1+2\cos\theta)^n\cos n\theta\,d\theta}{3+2\cos\theta} = \frac{2\pi}{\sqrt{5}}(3-\sqrt{5})^n,$$

(n is a positive integer).

<u>Theorem 4</u>.- Let $Q(Z)$ have a simple pole at $Z=a$ on the real axis, otherwise $Q(Z)$ satisfies the conditions of theorem

three, (with the necessary modification). Then

$$P\int_{-\infty}^{+\infty} Q(z)\,dz = 2\pi i \sum R^{+} + \pi i \ \operatorname{Res}_{a}\{Q(z)\},$$

where $\sum R^{+} \equiv$ sum of the residues in imaginary $Z > 0$.

Proof: Let $R > \rho > 0$. Let C_R denote the semicircle $|Z| = R$, $0 \leq \arg Z \leq \pi$. Let γ denote the small semicircle $|Z - a| = \rho$, $0 \leq \arg(Z - a) \leq \pi$, with its center at $x = a$ and its radius ρ. Let $\Gamma_{R,\rho}$ be the contour shown in figure 3.

$$\Gamma_{R,\rho} = C_R + (-R, a-\rho) + \gamma + (a+\rho, R)$$

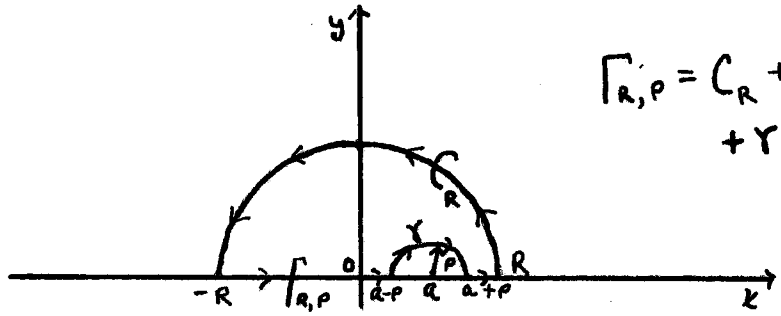

FIG. 3

Let R be large enough so that $\Gamma_{R,\rho}$ contains all the poles of $Q(Z)$ in imaginary Z greater than zero. Then the integral round $\Gamma_{R,\rho}$ tends to zero as $R \longrightarrow +\infty$, as before. We therefore have, if the path of integration is as indicated in figure 3,

$$\int_{\Gamma_{R,\rho}} Q(z)\,dz = 2\pi i \sum R^{+}.$$

Now observe

$$\int_{\Gamma_{R,\rho}} Q(z)\,dz = \int_{-R}^{a-\rho} Q(z)\,dz + \int_{\gamma} Q(z)\,dz + \int_{a+\rho}^{R} Q(z)\,dz + \int_{C_R} Q(z)\,dz,$$

by Cauchy's integral theorem.

As $R \longrightarrow +\infty$ and $\rho \longrightarrow 0$, $\displaystyle\int_{-R}^{a-\rho} Q(z)\,dz + \int Q(z)\,dz =$

$$P\int_{-\infty}^{\infty} Q(Z)dZ,$$

and

$$\lim_{R\to\infty}\int_{C_R} Q(Z)dZ = 0.$$

We must now consider

$$\int_{\gamma} Q(Z)dZ.$$

Since $Q(Z)$ has a simple pole at $Z = a$, $Q(Z) = \phi Z/(Z - a)$, where $\phi(Z)$ is regular at $Z = a$. Then

$$\int_{|z-a|=\rho} \phi(Z)/(Z - a)dZ = 2\pi i\phi(a).$$ We want to show that

$$\int_{\gamma} Q(Z)dZ = -\pi i\,\phi(a) \text{ as } \rho \longrightarrow 0.$$

Consider

$$\int_{\gamma} Q(Z)dZ + \pi i\phi(a)$$. Let $\epsilon > 0$ be arbitrary. Set $Z - a = \rho e^{i\theta}$. Then $dZ = i\rho e^{i\theta}d\theta$. Now

$$\int_{\gamma} Q(Z)dZ + \pi i\phi(a) = \int_{\gamma} \phi(Z)/(Z - a)dZ + \pi i\phi(a),$$

$$= \int_{\pi}^{0} \frac{\phi(a+\rho e^{i\theta})}{\rho e^{i\theta}}\, i\rho e^{i\theta}d\theta + \pi i\phi(a) = i\int_{\pi}^{0}\left[\phi(a+\rho e^{i\theta}) - \phi(a)\right]d\theta.$$

Now since Q(Z) is continuous at $Z = a$; there exist a ρ_0 depending upon ϵ such that

$$\left|\phi(a+\rho e^{i\theta}) - \phi(a)\right| < \epsilon \text{ , where } \rho < \rho_0. \text{ Hence}$$

$$\left|\int_{\gamma} Q(z)dz + i\pi\phi(a)\right| < \pi\epsilon$$

wherever $\rho < \rho_0$. It follows that

$$\int_{\gamma} Q(z)\,dz \longrightarrow -\pi i\phi(a), \text{ as } \rho \longrightarrow 0.$$

Since $\phi(a)$ is obviously the residue of $Q(Z) = \phi(Z)/(Z - a)$ at $Z = a$, we have therefore

$$P\int_{-\infty}^{\infty} Q(x)dx = 2\pi i \sum P^{+} + \pi i\phi(a). \text{ Q.E.D.}$$

Theorem 4 generalizes to:

Theorem 5.- Let Q(Z) have only a finite number of simple poles on imaginary Z equal to zero. Otherwise Q(Z) satisfies the conditions of theorem 3, (with necessary modifications). Then

$$P\int_{-\infty}^{\infty} Q(x)dx = 2\pi i \sum R^{+} + \pi i \sum R^{o},$$ where $\sum R^{o}$ denotes the sum of the residues of Q(Z) on imaginary Z equal zero.

Suppose $Q(Z) = N(Z)/D(Z)$, N, D are polynomials. Suppose further that the degree of D is greater than or equal to two plus the degree of N. If D(Z) has only simple poles on imaginary Z equal to zero, then the hypothesis of theorem 4 hold and hence the conclusion.

3. Evaluation of Infinite Integrals by Jordan's Lemma

We are now concerned with evaluating integrals of the type

$$\int_{-\infty}^{\infty} Q(x)\, e^{imx}\, dx,$$

where $m > 0$.

First we prove a very useful theorem which is usually referred to as Jordan's lemma.

Jordan's Lemma.- Let $Q(Z)dZ$ satisfy the following conditions:

(i) $Q(Z)$ is meromoric in the upper half-plane, with no poles on the real axis,

(ii) $Q(Z) \longrightarrow 0$ uniformly as $|Z| \longrightarrow +\infty$, $0 \leq \arg Z \leq \pi$,

(iii) is positive; then

$$\int_{C_R} e^{imz} Q(z)\, dz \longrightarrow 0 \text{ as } R \longrightarrow \infty,$$

where C_R denotes the semicircle $|Z| = R$, $0 \leq \arg Z \leq \pi$.

Proof: Set $Z = R\,e^{i\theta} = R\cos\theta + iR\sin\theta$. Then dZ becomes $iR\,e^{i\theta}\,d\theta$ and we have

$$\int_{C_R} e^{imz} Q(z)\, dz = iR \int_0^{\pi} e^{imR\cos\theta - mR\sin\theta}\, Q(Re^{i\theta})\, e^{i\theta}\, d\theta.$$

But

$$\left| \int_{C_R} e^{imz} Q(z)\, dz \right| \leq R \int_0^{\pi} e^{-mR\sin\theta} \left| Q(Re^{i\theta}) \right| d\theta.$$

By condition (ii) there exists, for any $\epsilon > 0$, an R_0 depending upon ϵ such that $|Q(Re^{i\theta})| < \epsilon$ for all $R > R_0$, and

$$\epsilon R\int_0^{\pi} e^{-mR\sin\theta}d\theta = 2\epsilon R\int_0^{\frac{\pi}{2}} e^{-mR\sin\theta}d\theta.$$

Now set $u(\theta) = \sin\theta/\theta$. Observe that $u(\theta) \longrightarrow 1$ as $\theta \longrightarrow 0$ and $u(\theta) \longrightarrow 2/\pi$ as $\theta \longrightarrow \pi/2$. We want to show that $u(\theta)$ decreases in $[0, \pi/2]$. Consider the derivative of $u(\theta)$. Thus

$u'(\theta) = \dfrac{\theta\cos\theta - \sin\theta}{\theta^2}$. Now set $h(\theta) = \theta\cos\theta - \sin\theta$, and consider $h'(\theta) = -\theta\sin\theta$, so that $h'(\theta) = -\theta\sin\theta \leq 0$ in $[0, \pi/2]$. Hence $h(\theta)$ cannot increase. But $h(0) = 0$, so that $u'(\theta) = h(\theta)/\theta^2 \leq 0$. Thus $u(\theta)$ cannot increase, and hence $u(\theta)$ decreases in $[0, \pi/2]$. Thus

$u(\theta) = \sin\theta/\theta \geq 2/\pi$ on $[0, \pi/2]$.

Hence

$$\left|\int_{C_R} e^{imz}Q(z)dz\right| < 2\epsilon R\int_0^{\frac{\pi}{2}} e^{-Rm\frac{2}{\pi}\theta}d\theta$$

$$= \frac{2\epsilon R\pi}{Rm2}\left[e^{\frac{-2mR\theta}{\pi}}\Big|_0^{\frac{\pi}{2}}\right] = \frac{\epsilon\pi}{m}\left[1 - e^{-mR}\right] < \frac{\epsilon\pi}{m},$$

for $(R > R_0)$. But ϵ is arbitary, hence the lemma. Q.E.D.

By virtue of this lemma and previous results we have the following theorem:

Theorem 6.- Let $Q(Z) = N(Z)/D(Z)$, where $N(Z)$ and $D(Z)$ are polynomials, and $D(Z) = 0$ has no root belonging to the real numbers, then if:

(i) the degree of $D(Z)$ exceeds that of $N(Z)$ by at least one,

(ii) $m > 0$,

$$\int_{-\infty}^{\infty} Q(x)\, e^{imx}\, dx = 2\pi i \sum R^{+},$$

where $\sum R^{+}$ denotes the sum of the residues of $Q(Z)e^{imz}$ at its poles in the upper half-plane.

If we write $f(Z) = Q(Z)e^{imz}$, we see that $f(Z)$ satisfies the conditions of Jordan's lemma and so

$\int_{\Gamma} f(Z)dZ \longrightarrow 0$ as $R \longrightarrow \infty$. On using the same contour as before, that is a large semicircle in the upper half-plane, and by letting $R \longrightarrow \infty$ we get

$$\int_{-\infty}^{\infty} Q(x)\, e^{imx}\, dx = 2\pi i \sum R^{+}. \qquad \text{Q.E.D.}$$

Example.- Prove that, if $a > 0$, $m > 0$,

$$\int_{0}^{+\infty} \frac{\cos mx\, dx}{(a^2+x^2)^2} = \frac{\pi}{4a^3}(1+am)e^{-am}$$

Proof: Recall $e^{imx} = \cos mx + i \sin mx$. Also observe that

$$\int_{0}^{+\infty} \frac{\cos mx}{(a^2+x^2)^2}\, dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\cos mx}{(a^2+x^2)^2}\, dx.$$

Moreover

$$R_e\left[\int_{-\infty}^{+\infty} \frac{e^{imx}}{(a^2+x^2)^2}\, dx\right] = \int_{-\infty}^{+\infty} \frac{\cos mx}{(a^2+x^2)^2}\, dx,$$

where

$R_e\left[\int_{-\infty}^{+\infty} \frac{e^{imx}}{(a^2+x^2)^2}\, dx\right]$, denotes the real part of

$$\left[\int_{-\infty}^{+\infty} \frac{e^{imx}}{(a^2+x^2)^2}\,dx\right].$$

Now observe

$Q(Z) = \dfrac{1}{(a^2 + Z^2)^2}$ is regular in imaginary $Z \geq 0$ except

for the pole $Z = ia$ (of order 2).

Note: $a^2 + Z^2 = 0$, $Z^2 = -a^2$ implies that $Z^2 = (ai)^2$ which implies that $Z = \pm ai$. Also $Q(Z) \longrightarrow 0$ uniformly as $|Z| \longrightarrow +\infty$. Hence for this $Q(Z)$ the conditions of our theorem are satisfied and therefore

$$\int_{-\infty}^{+\infty} \frac{e^{imx}}{(a^2+x^2)^2}\,dx = 2\pi i \sum R^+,$$

where $\sum R^+$ denotes the residues of $\dfrac{e^{imz}}{(a^2+z^2)^2}$ in the upper half-plane, that is, the residue at $Z = ai$. We calculate the residue at $Z = ai$ as follows:

$$\operatorname*{Res}_{ai}\left[\frac{e^{imz}}{(a^2+z^2)^2}\right] = \lim_{z\to ai}\left[\frac{d}{dz}\left\{\frac{(z-ai)^2 e^{imz}}{(a^2+z^2)^2}\right\}\right]$$

$$= \lim_{z\to ai}\left[\frac{d}{dz}\left\{\frac{e^{imz}}{(z+ai)^2}\right\}\right] = \lim_{z\to ai}\left[\frac{i(z+ai)^2 m e^{imz} - 2e^{imz}(z+ai)}{(ia+z)^4}\right]$$

$$= \frac{-i\,e^{-ma}(1+am)}{4a^3}.$$

Hence

$$\int_{-\infty}^{+\infty} \frac{e^{imx}}{(a^2+x^2)^2}dx = 2\pi i\left[\frac{-i e^{-ma}(1+am)}{4a^3}\right]$$

$$= \frac{\pi}{4a^3}(1+am)e^{-ma}.$$

Therefore

$$\int_0^{\infty} \frac{\cos mx}{(a^2+x^2)^2}dx = \frac{\pi}{4a^3}(1+am)e^{-ma}.$$

Exercise.- Prove

$$I = \int_0^{+\infty} \frac{\sin^2 mx}{x^2(a^2+x^2)} = \frac{\pi}{4a^3}\left(e^{-2ma} - 1 + 2ma\right).$$

4 Integrals Involving Many-Valued Functions[1]

A type of integral of the form

$$\int_0^{\infty} x^{a-1}Q(x)dx,$$

where a is not an integer, can also be evaluated by contour integration, but since Z^{a-1} is a many-valued function, it becomes necessary to use the cut plane. One method of dealing with integrals of this type is to use as a contour a large circle Γ, center at the origin, and radius R; but we must cut the plane along the real axis from 0 to $+\infty$ and also enclose the branch-point $Z = 0$ in a small circle γ of radius ρ. The contour is illustrated in figure 4.

Let a not equal an integer. Let Q(Z) be such that the

[1]E. G. Phillips, Functions of a Complex Variable With Applications (Interscience Publishers Inc., 1958), pp. 118-128

following conditions are satisfied:

(i) $Q(Z)$ has a finite number of poles in the plane, but no singular points on the real axis,

(ii) Z times $Z^{a-1}Q(Z) \longrightarrow 0$ uniformly both as $|Z| \longrightarrow 0$ and as $Z \longrightarrow \infty$. Then

$$\int_0^\infty x^{a-1} Q x \, dx = \frac{2\pi i \sum R}{1 - e^{2\pi i a}},$$

where $\sum R$ denotes the sum of the residues of $f(Z)$ inside the contour, $f(Z) = Z^{a-1}Q(Z)$.

Proof: Let our contour Γ be a large circle, center at the origin, and radius R; but cut along the real axis from 0 to ∞ and inclose the branch point $Z = 0$ in a small circle γ of radius ρ. Now since $Z\,f(Z) \longrightarrow 0$ unformily both as $|Z| \longrightarrow \infty$, and as $|Z| \longrightarrow 0$, we get the integral round Γ tending to zero as $R \longrightarrow \infty$ and the integral round γ tending to zero as $\rho \longrightarrow 0$; for on Γ, if R is large enough, $|Z\,f(Z)| < \epsilon$ and so $|f(Z)| < \epsilon/R$.

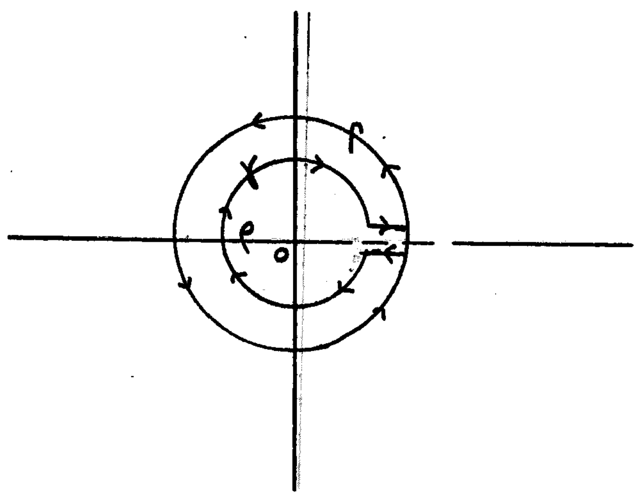

FIG. 4

Thus

$$\left| \int_\Gamma f(z)\,dz \right| < \frac{\epsilon}{R} 2\pi R = 2\pi\epsilon$$

Similarly on γ, $|Zf(Z)| < \epsilon$ if ρ is small enough, and so $|f(Z)| < \epsilon/\rho$ and

$$\left| \int_{\gamma} f(Z)\,dZ \right| < \frac{\epsilon}{\rho}\, 2\pi\epsilon$$

Hence on making $\rho \longrightarrow 0$ and $R \longrightarrow \infty$ we get

$$\int_0^\infty x^{a-1} Q x\, dx + \int_\infty^0 x^{a-1} e^{2\pi i (a-1)} Q(x)\, dx = 2\pi i \Sigma R,$$

where ΣR is the sum of the residues of $f(Z)$ inside the contour.

Observe that the values of x^{a-1} at points on the upper and lower edges of the cut are not the same, for, if $Z^{a-1} = r e^{i\theta}$, we have $Z^{a-1} = r^{a-1} e^{i\theta(a-1)}$ and the values of Z at points on the upper edge correspond to $|Z| = r$, $\theta = 0$, and at points on the lower edge they correspond to $|Z| = r$, $\theta = 2\pi$.

Since $e^{2\pi i (a-1)} = e^{2\pi i a}$, we get

$$\int_0^\infty x^{a-1} Q(x)\, dx = \frac{2\pi i \Sigma R}{1 - e^{2\pi i a}}.$$

Q.E.D.

Note: When calculating the residues at the poles, Z must be given its correct value $r^{a-1} e^{i\theta(a-1)}$ at each pole.

Example.- Prove that

$$\int_0^\infty \frac{x^{a-1}}{1+x}\, dx = \frac{\pi}{\sin a\pi}, \text{ if } 0 < a < 1.$$

Here we observe that, when $f(Z) = Z^{a-1}(1 + Z)^{-1}$, $Zf(Z)$ tends to zero as $|Z|$ tends to infinity, if $0 < a < 1$, and

$Z\,f(Z)$ tends to zero as $|Z|$ tends to zero, if $a > 0$. Hence, if $0 < a < 1$, the integral round Γ tends to zero as R tends to infinity and the integral round γ tends to zero as ρ tends to zero. Thus

$$\int_0^{\infty} \frac{x^{a-1}}{1+x}\,dx = \frac{2\pi i}{1-e^{2\pi i a}}\left\{\text{residue of } Z \quad (1 - Z)^{-1} \text{ at } Z = 1\right\}$$

At $Z = -1$ we have $r = 1$, $\theta = \pi$, and so,

$$\operatorname*{Res}_{-1}\left[\frac{z^{a-1}}{1+z}\right] = \lim_{z \to -1}\left\{(1+z)\,\frac{z^{a-1}}{1+z}\right\} = (-1)^{a-1} = e^{(a-1)\pi i} = -e^{a\pi i}.$$

Hence

$$\int_0^{\infty} \frac{x^{a-1}}{1+x}\,dx = -2\pi i\left\{\frac{e^{a\pi i}}{1-e^{2\pi i a}}\right\} = -2\pi i\,\frac{1}{e^{-a\pi i}-e^{a\pi i}} = \frac{\pi}{\sin a\pi}.$$

5. Expansion of a Meromorphic Function

We begin our discussion of the expansion of a meromorphic by considering the following theorem.

Theorem 7.- Let f be meromorphic. Let $a_1, a_2, \cdots$ be the simple poles of f and let $b_1, b_2, \cdots$ be the respective residues of f at the poles $a_1, a_2, \cdots$. Assume that

$$0 < |a_1| < |a_2| < |a_3| < \cdots.$$

Let $f(Z)/Z \longrightarrow 0$ uniformly as $Z \longrightarrow \infty$. Then

$$f(Z) = f(0) + \sum_{n=1}^{\infty}\left[\frac{b_n}{Z-a_n} + \frac{b_n}{a_n}\right].$$

Proof: Let C_n be a positively oriented closed path containing the origin and the points $a_1, a_2, \cdots, a_n$. Let R_n be the minimum distance from 0 to C_n (written: $R_n = \min d(0, C_n)$). Let $L_n =$ length of C_n. Note that $R_n \longrightarrow +\infty$ as $n \longrightarrow +\infty$.

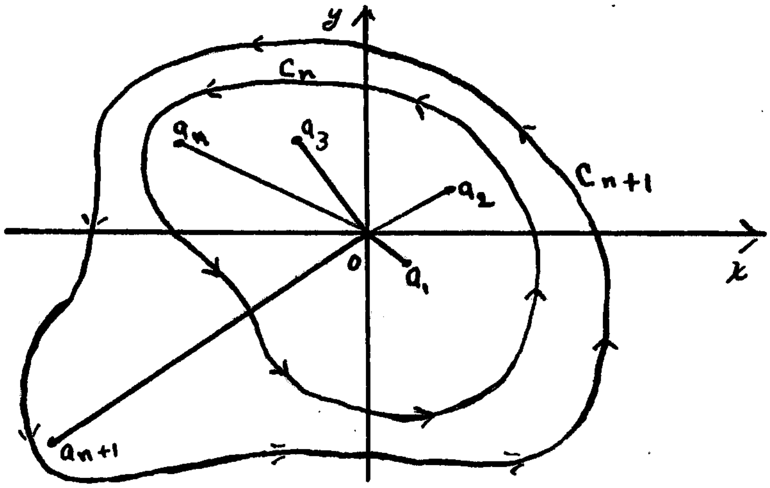

FIG. 5

Let $\epsilon > 0$ be given, then there exist an N depending upon ϵ such that $|f(Z_n)/Z_n| < \epsilon$ for all $n > N$. Here Z_n is any point of C_n which gives us the minimum R_n (see figure 5). Now consider the integral

$$J_n = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta(\zeta - z)}\,d\zeta ,$$

where Z is fixed for the moment and observe

$$J_n = \frac{-f(0)}{Z} + \frac{f(Z)}{Z} + \sum_{k=1}^{n} \frac{b_k}{a_k(a_k - z)}.$$

Note: The residue of $f(\zeta)/(\zeta(\zeta-z))$ at the origin is given by

$$\lim_{\zeta \to 0}\left\{\frac{\zeta[f(\zeta)]}{\zeta(\zeta - z)}\right\} = -\frac{f(0)}{z}.$$

The residue of $\frac{f(\zeta)}{\zeta(\zeta-z)}$ at $\zeta = Z$ is given by

$$\lim_{\zeta \to z}\left\{\frac{(\zeta - z)f(\zeta)}{\zeta(\zeta - z)}\right\} = \frac{f(z)}{z}.$$

If now we can show that $J \longrightarrow 0$ as $n \longrightarrow \infty$, the theorem is proved. Observe

$$|J_n| < \frac{1}{2\pi} \epsilon \; \frac{L_n}{R_n - |Z|},$$

for all $n > N$ since for $\epsilon > 0$ there exist an N depending upon ϵ such that $|f(Z_n)/Z_n| < \epsilon$ for all $n > N$, and the fact that

$$|\zeta - z| \geq |\zeta| - |z| \geq R_n - |z| \,.$$

But note that $L_n \leq 8R_n$, since $C_n \leq$ the perimeter of a square with sides equal to $2R_n$.

$$|J_n| \leq \frac{1}{\pi} \epsilon \; \frac{4\, R_n}{R_n - |Z|}$$

Hence $J_n \longrightarrow 0$ as $n \longrightarrow +\infty$, since ϵ is arbitary. Therefore

$$f(Z) = f(0) + \sum_{n=1}^{\infty} \frac{z\, b_n}{(z - a_n)a_n},$$

$$= f(0) + \sum_{n=1}^{\infty} \left[\frac{b_n}{(z - a_n)} + \frac{b_n}{a_n} \right].$$

Q.E.D.

<u>Example</u>.- Using theorem 7, we prove that

$$f(Z) = \csc Z = 1/Z + \sum_{n=1}^{\infty} \frac{2z(-1)^n}{z^2 - n^2\pi^2}.$$

Proof: $f(Z) = \csc Z = 1/\sin Z$ has simple poles at $Z_n = n\pi$, $n = 0, \pm 1, \pm 2, \cdots$, since in the Laurent expansion of $f(Z)$ we have

$$f(Z) = \cdots + \frac{a_{-1}^k}{(z - z_k)} + a_0^k + a_1^k(z - z_k) + \cdots,$$

and

$$\lim_{z \to n\pi}\left[(z-n\pi)\cdot\frac{1}{\sin z}\right] = \lim_{z \to n\pi}\left[\frac{1}{\cos n\pi}\right] = \frac{1}{\cos n\pi} = (-1)^n$$

thus the pole at the origin is a simple pole. The theorem cannot be applied until we eliminate the pole at the origin. We do this in the following manner. Observe

$$\csc z = \frac{1}{\sin z} = \frac{1}{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}$$

$$= \frac{1}{z}\left(1 + \frac{z^2}{3!} - \frac{z^4}{5!} + \cdots\right)$$

$$= \frac{1}{z} + \frac{z}{3!} + \cdots.$$

Thus, $\csc Z - 1/Z = Z/3! + \cdots$. Set $g(Z) = \csc Z - 1/Z$ and observe that $g(Z)$ has no pole at the origin. The poles of $g(Z)$ are at $Z_n = n\pi$, $n = \pm 1, \pm 2, \pm \cdots$. Now we want to find the residues of $g(Z)$ at Z_n. They are given by

$$\lim_{z \to n\pi}\left[(z-n\pi)\cdot\frac{z - \sin z}{z \sin z}\right] = \lim_{z \to n\pi}\left[\frac{(z-n\pi)(1-\cos z)+(z-\sin z)}{(z\cos z + \sin z)}\right]$$

$$= \frac{n\pi}{n\pi \cos n\pi} = \frac{1}{\cos n\pi} = (-1)^n.$$

Now applying theorem 7:

Since $g(Z)/Z = \csc Z/Z - 1/Z^2 = 1/Z \sin Z - 1/Z^2 \longrightarrow 0$ uniformly as $|Z| \longrightarrow +\infty$, we have

$$g(Z) = \csc Z - 1/Z = g(0) + \sum_{n=-\infty}^{\infty}{}' \left[\frac{(-1)^n}{z - n\pi} + \frac{(-1)^n}{n\pi} \right],$$

(where $\sum'$ indicates that $n = 0$ is omitted in the summation).

But $g(0) = 0$ since

$$\lim_{z\to 0}\left(\csc z - \frac{1}{z}\right) = \lim_{z\to 0}\left(\frac{z - \sin z}{z \sin z}\right) = \lim_{z\to 0}\left(\frac{1 - \cos z}{z \cos z + \sin z}\right) = 0.$$

so that

$$g(Z) = \sum_{n=-\infty}^{\infty}{}' \left[\frac{(-1)^n}{z - n\pi} + \frac{(-1)^n}{n\pi} \right]$$

$$= \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{z - n\pi} + \frac{(-1)^n}{n\pi} \right] + \sum_{n=-\infty}^{-1} \left[\frac{(-1)^n}{z - n\pi} + \frac{(-1)^n}{n\pi} \right]$$

$$= \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{z - n\pi} + \frac{(-1)^n}{n\pi} \right] + \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{z + n\pi} - \frac{(-1)^n}{n\pi} \right]$$

$$= \sum_{n=1}^{\infty} \left[\frac{z(-1)^n + n\pi(-1)^n + z(-1)^n - n\pi(-1)^n}{z^2 - n^2\pi^2} \right]$$

$$= \sum_{n=1}^{\infty} \frac{2z(-1)^n}{z^2 - n^2\pi^2}.$$

Therefore

$$f(z) = \csc z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z(-1)^n}{z^2 - n^2\pi^2}.$$

Exercise.- Prove that

$$\sec Z = 4\pi \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)}{(2n+1)^2\pi^2 - 4Z^2}.$$

6. Summing Certain Infinite Series by the Calculus of Residues

The method of contour integration can be used with advanage for summing series of the type $\sum f(n)$, if f is a meromorphic function of a fairly simple kind. We now prove the following theorem.

Theorem.- Let f be a rational function such that Z f(Z) tends to zero uniformly as $|Z|$ tends to $+\infty$. Let f(Z) have poles at $a_1, a_2, \cdots, a_p$ with residues $\ell_1, \ell_2, \cdots, \ell_p$ respectively. Then

$$\text{(i)} \quad \sum_{n=-\infty}^{+\infty} f(n) = -\sum_{k=1}^{p} \pi \ell_k \cot \pi a_k,$$

$$\text{(ii)} \quad \sum_{n=-\infty}^{+\infty} (-1)^n f(n) = -\sum_{k=1}^{p} \pi \ell_k \csc \pi a_k.$$

Proof: (i) Let C_n be a simple closed path containing the origin but not passing through any integral values, such that $R_n = \min d(0, C_n) \longrightarrow +\infty$, as $n \longrightarrow +\infty$. Now consider the integral

$$J = \frac{1}{2\pi i} \int_{C_n} f(z)\, \pi \cot \pi z \, dz.$$

Note: Cot π Z has simple poles at $Z = k$, $k = 0, \pm 1, \cdots$, and the residues of cot π Z are calculated as follows:

$$\operatorname*{Res}_{z=k}\left[\cot \pi z\right] = \lim_{z\to \pi k}\left[(z-\pi)\cot \pi z\right] = \lim_{z\to k}\left[\frac{(z-k)(\cos \pi z)}{\sin \pi z}\right]$$

$$= \lim_{z\to k}\left[\frac{-\pi(z-\pi)\sin \pi z + \cos \pi z}{\pi \cos \pi z}\right]$$

$$= \frac{1}{\pi}.$$

Now by the residue theorem

$$J = \sum_{a_k \varepsilon C_n} \pi b_k \cot \pi a_k + \sum_{k \varepsilon C_n} f(k).$$

We want to show that $J \longrightarrow 0$ as $n \longrightarrow +\infty$. Let $\epsilon > 0$ be given and arbitrary, then there exist an N depending upon ϵ such that

$|Z_n f(Z_n)| < \epsilon$, for all $n > N$. Here Z_n is any point of C_n

which gives us the minimum R_n. Thus

$$|J| = \frac{1}{2\pi}\left|\int_{C_n}\left[z f(z)\right]\frac{\pi \cot \pi z}{z}\, dz\right| < \frac{\pi}{2\pi}\epsilon M \frac{L_n}{R_n} \leq \frac{1}{2}\epsilon M \frac{8R_n}{R_n},$$

where M is the upperbound of $\cot \pi Z$ on C_n and L_n is the length of C_n, for all $n > N$. Now since ϵ is arbitrary $J \longrightarrow 0$ as $n \longrightarrow +\infty$. Hence

$$\sum_{k=-\infty}^{\infty} f(k) = -\sum_{k=1}^{P} \pi b_k \cot a_k.$$

Now (ii) may be proved similarly.

<u>Example</u>.- Find the sum of the series $\displaystyle\sum_{n=0}^{+\infty} \frac{(-1)^n}{n^2+a^2}$

Solution: $f(Z) = 1/(Z^2 - a^2)$. Note that $Z\, f(Z) \longrightarrow 0$ uniformly as $|Z| \longrightarrow +\infty$. $f(Z)$ is a rational function, with poles at $Z = \pm a i$. Thus we see that theorem 8, (ii) is applicable. Recall:

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i},$$

thus

$$\sin a i = \frac{e^{-a} - e^{a}}{2i} = i\left(\frac{e^{a} - e^{-a}}{2}\right) = i \sinh a.$$

We calculate the residues of $\csc \pi Z$ at $Z = \pm a\, i$ as follows:

$$\operatorname*{Res}_{z=ai}\left[\frac{1}{z^2 - a^2}\right] = \frac{1}{z + ai}\Bigg|_{z=ai} = \frac{1}{2ai}.$$

$$\operatorname*{Res}_{z=-ai}\left[\frac{1}{z^2 - a^2}\right] = \frac{1}{z - ai}\Bigg|_{z=-ai} = -\frac{1}{2ai}.$$

Hence

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 + a^2} = -\left[\frac{\pi}{2ai}\csc \pi a i + \frac{\pi}{2ai}\csc \pi a i\right]$$

$$= -\left[\frac{\pi}{ai}\cdot\frac{1}{\sin \pi a i}\right] = \frac{\pi}{a \sinh \pi a}.$$

But observe

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 + a^2} = \frac{1}{a^2} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2 + a^2}$$

$$= -\frac{1}{a^2} + 2\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2 + a^2}.$$

Hence

$$-\frac{1}{a^2} + 2\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+a^2} = \frac{\pi}{a} \operatorname{csch} \pi a.$$

Finally

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+a^2} = \frac{1}{2a^2} + \frac{\pi}{2a} \operatorname{csch} \pi a.$$

CHAPTER II

THE INVERSE THEOREM FOR ANALYTIC FUNCTIONS

7. Poles and Zeros of Meromorphic Functions

We begin our discussion of the inverse theorem for analytic functions by recalling the definition of a meromorphic function.

Definition.- A function f whose only singularities in the finite plane are poles, is called a meromorphic function.

We re-state theorem 2, in a slightly different form and prove it by making use of the variation of the logarithm of f(Z), written log f(Z), around a specific contour C.

Theorem 9.- Let f be meromorphic in a bounded region G and let f be regular on the boundary C of G and not equal to zero on C. Then

$$N - P = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz,$$

where N is the number of zeros and P the number of poles of f inside C. (A pole of order m must be counted m times).

Proof: (i) Suppose $Z = a$ is a zero of order m, then, in the neighborhood of this point $f(Z) = (Z - a)^m \phi(Z)$, where $\phi(Z)$ is regular and not zero. Hence

$$\frac{f'(z)}{f(z)} = \frac{m}{z-a} + \frac{\phi'(z)}{\phi(z)}.$$

Since the last term is regular at $Z = a$, we see that $f'(Z)$ divided by $f(Z)$ has a simple pole at $Z = a$ with residue m.

Similarly, if $Z=b$ is a pole of order k, $f'(Z)/f(Z)$ has a simple pole at $Z=b$ with residue $-k$. It follows, by the residue theorem that

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N - P.$$

(ii) Suppose $f(Z)$ is regular throughout G. Then

$$N = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz \quad , \text{ since } P = 0.$$

Set $d/(dZ) \cdot \log f(Z) = f'(Z)/f(Z)$. Then

$$N = \frac{1}{2\pi i}\int_C d \log f(z),$$

which may be written as

$$N = \frac{1}{2\pi i}\Delta_C \log f(z),$$

where $\Delta_C \log f(Z)$ reads: variation of the $\log f(Z)$ around the contour C. But $\log f(Z) = \log|f(Z)| + i \arg f(Z)$, where $\arg f(Z)$ denotes the argument of $f(Z)$. Hence

$$N = \frac{1}{2\pi i}\Big[\Delta_C \log|f(z)| + \Delta_C\, i \arg f(z)\Big] = \frac{1}{2\pi}\Delta_C \arg f(z),$$

since $\log|f(Z)|$ is one-valued. Therefore

$$N = \frac{1}{2\pi}\Delta_C \arg f(z)$$

This result is known as the principle of the argument. Q.E.D.

8. Rouche's Theorem

We now state and prove Rouche's theorem.

Theorem 10.- Let f and g be regular functions inside a simple closed path C and let $f(Z) \neq 0$ on C. Let $|f(Z)| > |g(Z)|$ on C. Then the number of zeros of f is equal to the number of f + g inside C.

Proof: Recall $N = 1/2\pi \, \Delta_c \arg f(Z)$, where N is the number of zeros of f inside C. We want to show that $2\pi N_{f+g}$ (the number of zeros of f(Z) + g(Z) in C) equals $2\pi N_f$ (the number of zeros of f(Z) in C).

Observe

$$2\pi N_{f+g} = \Delta_c \arg (f+g) = \Delta_c \arg \left[f\left(1+\frac{g}{f}\right)\right]$$
$$= \Delta_c \left[\arg f + \arg\left(1+\frac{g}{f}\right)\right]$$
$$= \Delta_c \arg f + \Delta_c \arg\left(1+\frac{g}{f}\right)$$
$$= 2\pi N_f + \Delta_c \arg\left(1+\frac{g}{f}\right).$$

Now we want to show that $\Delta_c \arg (1 + g/f) = 0$.

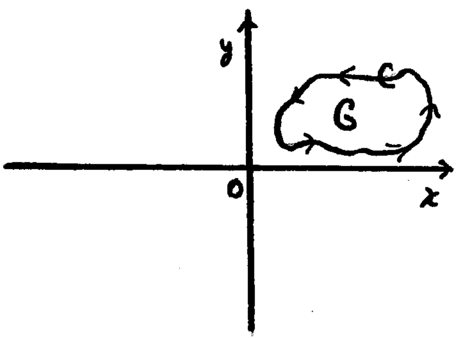

FIG. 6

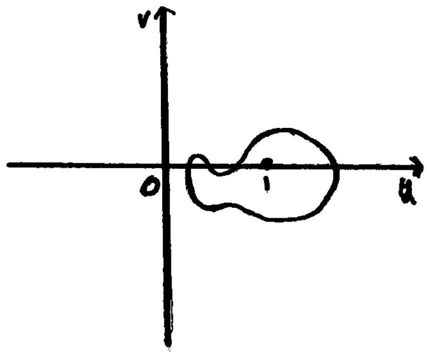

FIG. 7

Since $|g(Z)/f(Z)| < 1$ on C in the Z-plane, we have that

$$w = 1 + g(z)/f(z)$$

lies inside the circle $|w - 1| < 1$ in the w-plane, (see figures 6 and 7). Hence as we go around the curve C in the Z-plane the path traced by $w = 1 + g(Z)/f(Z)$ cannot encircle the origin in the w-plane. Thus we have that

$$\Delta_c \arg\left[1 + g(z)/f(z)\right] = 0$$

Therefore $N_{f+g} = N_f$. Q.E.D.

We can now prove the very important inverse theorem for analytic functions.

9. <u>The Inverse Theorem for Analytic Functions</u>

<u>Theorem</u> 11.- Let f be regular in a region G, and $f'(Z_0) \neq 0$ for some Z_0 belonging to G. Then there exist positive number η and ρ such that the values of f in $|f(Z) - f(Z_0)| < \eta$ are taken on once and only once for all Z belonging to $|Z - Z_0| < \rho$.

Proof: There exist a δ such that $f(Z) \neq 0$ in $0 < |Z - Z_0| < \delta$. Moreover, There exist a $\rho < \delta$ such $f(Z) \neq 0$ in $|Z - Z_0| \leq \rho$. Let η be the $\min\limits_{|z-z_0|<\rho} |f(Z) - F(Z_0)|$. Then $|f(Z) - f(Z_0)| \geq \eta$ on $|Z - Z_0| = \rho$. Set $w_0 = f(Z_0)$, and $w = f(Z)$. We want to show now that if w_1 belongs $|w - w_0|$, then this value is taken on once and omly once provided Z_1 belongs $|Z - Z_0| < \rho$. Observe that $w_1 - w_0$ is a complex number with the property $|w_1 - w_0| < \eta$. But note

$$f(z) - w_1 = f(z) - w_0 + w_0 - w_1.$$

Since $|f(Z) - w_0| > |w_0 - w_1|$ on $|Z - Z_0| = \rho$, we have by Rouché's theorem that $f(Z) - w_1$ has the same number of zeros inside $|Z - Z_0| = \rho$ as does $f(Z) - w_0$, which has precisely one. Q.E.D.

Theorem 11 enables us to define the inverse function for an analytic function. That is to say we can define the inverse function $Z = \phi(w)$, defined in $|w - w_0| < \eta$ such that $f(z) = f[\phi(w)] = w$.

See figure 8.

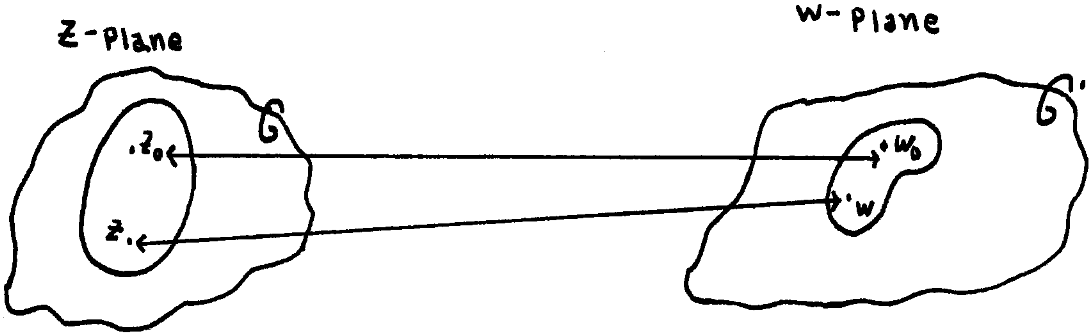

FIG. 8

<u>Theorem</u> 12.- Let ϕ be regular in $|w - w_0| < \eta$. Then

$\phi'(w) = \dfrac{1}{f'(Z)}$ for w belonging to $|w - w_0| < \eta$.

Proof: Let $w_1 = f(Z_1)$, where Z_1 belongs to $|Z - Z_0| < \rho$.

Then

$$\phi'(w) \stackrel{d}{=} \lim_{w_1 \to w} \frac{\phi'(w_1) - \phi(w)}{w_1 - w}$$

$$= \lim_{z_1 \to z} \frac{z_1 - z}{f(z_1) - f(z)}$$

$$= \lim_{z \to z_1} \frac{1}{\dfrac{f(z_1) - f(z)}{z_1 - z}} = \frac{1}{f'(z)}.$$

Q.E.D.

CHAPTER III

CONFORMAL REPRESENTATION

10. Mapping[1]

Properties of a real-valued function f of a real variable x are exhibited geometrically by the graph of the function. The equation $y=f(x)$ establishes a correspondence between points x on the x axis and points y on the y axis; that is, it maps points x into points y. The graphical description is improved by mapping each point x into a point (x,y) of the xy-plane at a directed distance y above or below point x. The curve that is obtained is the graph of f. In a similar way we use a surface to exhibit graphically a real-valued function f of the real variables x and y.

But when $w=f(Z)$ and the variables w and Z are complex, no such convenient graphical representation of the function f is available, because a plane is needed for the representation of each variable. Some information can be displayed graphically, however, by showing sets of corresponding points Z and w. It is generally easer to draw separate complex planes for the two planes Z and w. Then correspomding to each point (x,y) in the Z-plane, in the domain of definition G of f, there is a point (u,v) in the w-plane belonging to the range G' of the function

[1]Ruel V. Churchill, Complex Variables and Applications (Second Ed.; New York: McGraw-Hill Book Co., 1960), pp 20-21.

f, where $w = u + iv$.

Definition.- If to each point Z of a region G of the Z-plane, called the domain, there corresponds a unique point $f(Z) = w$ of a region G' of the w-plane, called the range, then there is said to be a mapping or map f of the region G into the region G' and the point $w = f(Z)$ is said to be the image of the point Z.

Suppose Z goes over to w under the mapping f, that is $f(Z) = w$ and w goes over to σ under the mapping g, or $g(w) = \sigma$, that is

$$Z \xrightarrow{f} w \text{ and } w \xrightarrow{g} \sigma,$$

then Z goes over to σ under the composite mapping g(f), or $g[f(Z)] = \sigma$, see figure 9.

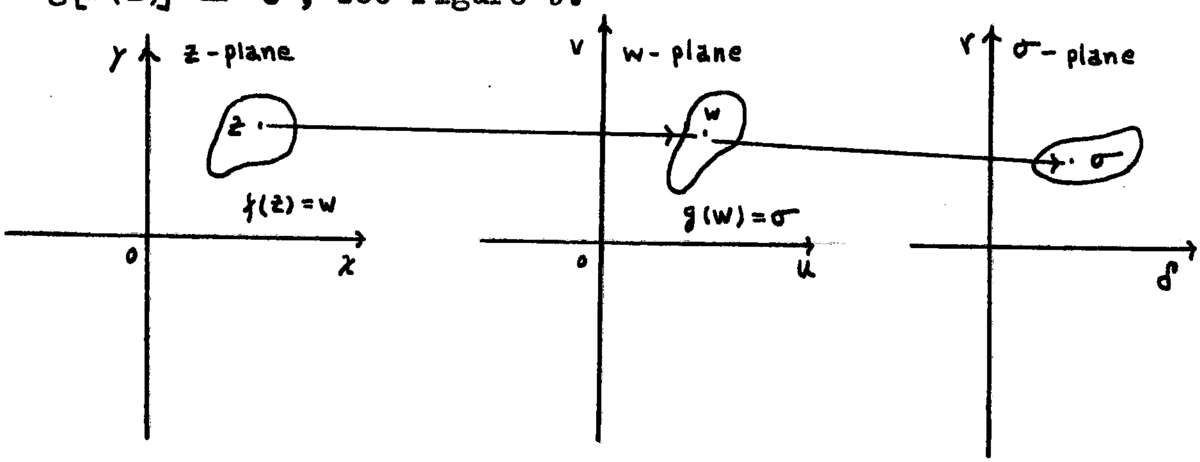

FIG. 9

Note: $g[f(Z)]$ does not necessarily equal $f[g(Z)]$.

Using different notation the mappings or transformations above may be stated as follows: if $w = f(Z) = TZ$ and $\sigma = g(w) = SW$, then $\sigma = S(TZ) = STZ$. This defines the composite transformation which takes Z into σ.

11. Isogonal and Conformal Transformations

Suppose $w = f(Z)$ is analytic at a point Z_0 of a region G of the Z-plane, and C_1 and C_2 are two continuous curves passing through the point Z_0. Let the tangents to the curves C_1 and C_2 at the point Z_0 make angles α_1, α_2, with the real axis, and suppose that $f'(Z_0) \neq 0$. We want to find the mapping of this figure on the w-plane.

Let Z_1 and Z_2 be points on the curves C_1 and C_2 near to Z_0 at the same distance r from Z_0, so that $Z_1 - Z_0 = r e^{i\theta_1}$ and $Z_2 - Z_0 = r e^{i\theta_2}$, then as $r \longrightarrow 0$, $\theta_1 \longrightarrow \alpha_1$ and $\theta_2 \longrightarrow \alpha_2$.

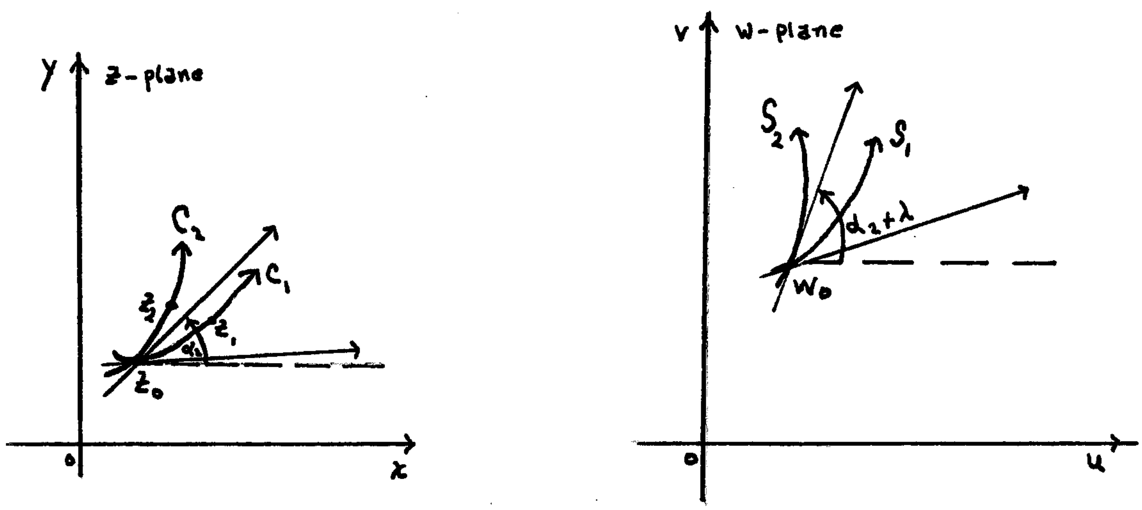

FIG. 10

The point Z_0 goes over to w_0 in the w-plane and Z_1 and Z_2 go over to points w_1 and w_2 which describe curves S_1 and S_2. Let

$$w_1 - w_0 = \rho_1 e^{i\phi_1}, \quad w_2 - w_0 = \rho_2 e^{i\phi_2}.$$

Then, by the definition of a regular function,

$$\lim_{z_1 \to z_0} (w_1 - w_0)/(Z_1 - Z_0) = f'(Z_0),$$

and since the right-hand side is not zero, we may write it as $Re^{i\lambda}$. We have then

$$\lim_{r\to 0} \rho_1 e^{i\phi_1}/re^{i\theta_1} = Re^{i\lambda},$$

and so $\lim (\phi_1 - \theta_1) = \lambda$ or $\lim \phi_1 = \alpha_1 + \lambda$.

Thus we see that the curve S_1 has a definite tangent at w_0 making an angle $\alpha_1 + \lambda$ with the real axis.

Similarily, S_2 has a definite tangent at w_0 making an angle $\alpha_2 + \lambda$ with the real axis.

It follows that S_1 and S_2 cut at the same angle as the curves C_1 and C_2. Further, the angle between the curves S_1 and S_2 has the same sense as the angle between the curves C_1 and C_2. We now define conformal and isogonal mappings.

Definitions.- (i) An isogonal mapping is a mapping which preserves magnitudes of angles but not necessarily the sense of rotation.

(ii) A conformal mapping is a mapping which preserves both the magnitudes of angles and the sense of rotations.

Thus we see that the regular function f, for which $f'(Z_0) \neq 0$, determines a conformal transformation. A point at which $f'(Z_0)$ is zero is called a critical point of the function f.

Now that we have acquired the concept of mapping, or transformation of points, by a function f of a complex variable Z, we shall apply this concept to particular types of functions.

12. Linear Functions

(i) The most simple example of a conformal mapping is the identity mapping

$$w = f(Z) = Z.$$

(ii) The most simple after (i) is the linear mapping

$$w = f(Z) = Z + \alpha ,$$

where α is a complex constant. This mapping is a translation of every point Z through the vector representing C. That is, if $Z = x + iy$, $w = u + iv$ and $C = C_1 + iC_2$, then the image of any point (x,y) in the Z-plane is the point $(x + C_1, y + C_2)$ in the w-plane. Since every point in any region of the Z-plane is mapped upon the w-plane in this same manner, the image of the region is simply a translation of the given region. The two regions have the same shape, size and orientation.

Geometrically this is clear, but analytically we proceed as follows: for the straight line $y = mx + b$,

$$w = u + iv \text{ and } Z + \alpha = x + \alpha_1 + i(y + \alpha_2).$$

So $u + iv = (x + \alpha_1) + i(y + \alpha_2)$. Hence

$$u = x + \alpha_1$$

$$v = y + \alpha_2.$$

Therefore, $y = mx + b \longrightarrow (v - \alpha_2) = m(u - \alpha_1) + b$, that is, $v = mu + (\alpha_2 + b - m\alpha_1)$. This mapping takes the circle $|Z - Z_0| = r$ into the circle $|w - \alpha - Z_0| = r$, or $|w - (Z_0 + \alpha)| = r$.

<u>Exercise</u>.- Prove that for the translation $w^2 = (Z - \alpha)$ times $(Z - \beta)$, the critical points are $Z = \alpha$, $Z = \beta$, $Z = 1/2$ $(\alpha + \beta)$, $w = 0$ and $w = \pm 1/2\ i(\alpha - \beta)$.

(iii) Let be a complex constant whose polar form is

$$\beta = b e^{i\lambda}.$$

Then, if

$$Z = r e^{i\theta},$$

the function

$$w = \beta Z = b r e^{i(\theta+\lambda)}$$

maps the point (r,θ) in the Z-plane into that point in the w-plane whose polar coordinates are br, $\theta + \lambda$. That is , the mapping consists of a rotation of the radius vector of the point Z about the origin through the angle $\lambda = \arg \beta$ and an expansion or contraction of the radius vector by the factor $b = |\beta|$. Every region in the Z-plane is transformed by this rotation and expansion into a geometrically similar region in the w-plane.

Consider the transformation

$$w = \sigma Z, |\sigma| = 1,$$

where λ is a complex number. The above is a trivial case of $w = \beta Z$, $\beta \neq 0$ and β a complex number. Observe $dw/dZ = \beta \neq 0$. Thus $w = \beta Z$ is conformal for all points in the Z-plane.

As a further illustration consider the straight line

$$(1)\ Ax + Cy + D = 0.$$

Observe that $x + iy = Z = 1/(\beta)\, w$, since $w = \beta Z$.

$$\frac{1}{\beta w} = \frac{\bar{\beta}}{|\beta|^2} w = \frac{(\beta_1 - i\beta_2)}{|\beta|^2} (u + iv)$$

where $\beta = \beta_1 + i \beta Z$, $w = u + iv$ and $\bar{\beta}$ equals the conjugate of β. But

$$\frac{\beta_1 - i\beta_2}{|\beta|^2}(u + iv) = \frac{-1}{|\beta|^2}\left[(\beta_1 u + \beta_2 v) + i(\beta_1 v - \beta_2 u)\right]$$

Hence

$$x = 1/|\beta|^2(\beta_1 u + \beta_2 v)$$

and $$y = 1/|\beta|^2(\beta_1 v - \beta_2 u)$$

Now upon substituting in (1) we have

$$A(\beta_1 u + \beta_2 v) + C(\beta_1 v - \beta_2 u) + |\beta|^2 D = 0,$$

or

$$(A\beta_1 - C\beta_2)u + (A\beta_2 + C\beta_1)v + |\beta|^2 D = 0,$$

which is the straight line obtained under the rotation

$$w = \beta Z.$$

The function $f(Z) = w = \beta Z$ is sometimes referred to as a rotational contraction or expansion according as $|\beta| < 1$ or $|\beta| > 1$ If $|\beta| = 1$, then $w = \beta Z$ is a pure rotation.

Note: The circle C consisting of the set of all points Z such that $|Z - Z_0| = r$ with center at Z_0 and radius r, is transformed by

$$w = \beta Z$$

into the circle C consisting of the set of all points w such that $|w/(\beta) - Z_0| = r$, with center at Z_0 and radius r or

$$C = \{\omega \mid |\tfrac{\omega}{\beta} - z_0| = r\}$$

(read the circle C consisting of the set of all points w such that $|w - \beta Z_0| = |\beta| r$) with center at βZ_0 and of radius $|\beta| r$.

(iv) If we write the transformation $w = Z + \alpha$ as $w = TZ$ and write $w = \beta Z$ as $w = SZ$, then the transformation

$$L = TS$$

is the most general linear transformation. Observe that

$$LZ = TSZ = T(SZ) = T\beta Z = \beta Z + \alpha,$$

and is therefore a conformal napping. The transformation

$$L' = ST$$

read L prime equals ST is also a linear transformation, since

$$L'Z = STZ = S(TZ) = S(Z + \alpha) = \beta (Z + \alpha) = \beta Z + \beta\alpha .$$

Thus we see that

$$L' \neq L.$$

The mapping L is conformal for all Z, provided $\beta \neq 0$. Moverover L takes straight lines into straight lines and takes circles into circles. The general mapping

$$L = \beta Z + \alpha$$

consists of a rotation through the angle arg β and a magnification by the factor $|\beta|$, followed by a translation through the vector α . As an illustration consider the following example.

Example.- Find the image of the rectangle with vertices $Z_o = (0 + 0)$, $Z_1 = (0 + 2i)$, $Z_2 = (1 + 2i)$ and $Z_3 = (1 + 0)$ under the transformation

$$w = (1 + i)Z_k + 2 - i, \ k = o, 1, 2, 3.$$

Show the region graphically.

Solution: Observe that in this case $\beta = (1 + i)$ and $\alpha = (2 - i)$. Applying this transformation to each of the given points in the Z-plane we obtain the desired corresponding set of points in the w-plane

$$w_o = \beta Z_o + \alpha = (1 + i)(0 + 0) + (2 - i) = (2 - i)$$
$$w_1 = \beta Z_1 + \alpha = (1 + i)(0 + 2i) + (2 - i) = (0 + i)$$
$$w_2 = \beta Z_2 + \alpha = (1 + i)(1 + 2i) + (2 - i) = (1 + 2i)$$
$$w_3 = \beta Z_3 + \alpha = (1 + i)(1 + 0) + (2 - i) = (3 + 0).$$

Now observe, arg $(1 + i)\ \pi/4$ and $|1 + i| = \sqrt{2}$. Therefore

$$w = (1 + i)Z_k + (2 - i),\ k = 0,\ 1,\ 2,\ 3,$$

transforms the rectangle with vertices $Z_0 = (0 + 0)$, $Z_1 = (0 + 2i)$, $Z_2 = (1 + 2i)$ and $Z_3 = (1 + 0)$ into the rectangle with the vertices $w_0 = (2 - i)$, $w_1 = (0 + i)$, $w_2 = (1 + 2i)$ and $w_3 = (3 + 0)$, see figure 10.

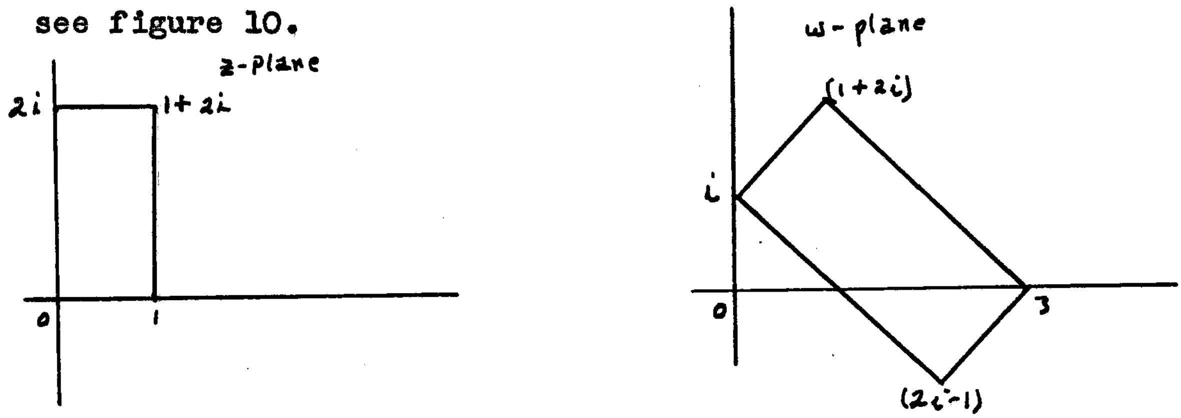

FIG. 11

13. The Function w Z^n[1]

First we see that the image of any point (r,θ) is that point in the w-plane whose polar coordinates are

$$\rho = r^2,\quad \phi = 2\theta$$

Where we consider the particular case for n   2, and describe the transformation in terms of polar coordinates by setting

$$Z = re^{i\theta} \text{ and } w = \rho e^{i\phi},\ \text{then } \rho e^{i\phi} = r^2 e^{2i\theta}$$

In particular the function Z^2 maps the entire first quadrant of the Z-plane, $0 \leq \theta \leq \pi/2$, $r \geq 0$, upon the entire upper half plane of the Z-plane(see figure 12).

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[1] Ibid, pp 67-68.

Circles about the origin in the Z-plane with radius r_o are transformed into circles about the origin with radius r_o^2. The semicircular region $r \leq r_o$, $0 \leq \theta \leq \pi$ is mapped onto the circular region $\rho \leq r_o^2$, and the first quadrant of that semicircular region is mapped onto the upper half of the circular region as indicated in figure 12 by the broken lines.

In each of the above mappings of regions by the transformations $w = Z^2$, there is just on point in the transformed region corresponding to a given point in the original region and conversely; thatis, there is a unique one to one correspondence between points in the two regions. This uniqueness does not exist, however, for the circular region $r \leq r_o$, $0 \leq \theta \leq 2\pi$, and its image $\rho \leq r_o^2$, since each point w of the latter region is the image of two points Z and -Z of the former.

In rectangular coordinates the transformation $w \quad Z^2$ becomes

$$u + iv = (x + iy)^2 = x^2 - y^2 + 2xyi,$$

then

$$u = x^2 - y^2 \text{ and } v = 2xy.$$

If imaginary Z equals y equals zero (the equation of the real axis), then $u = x^2$ and $v = 0$, so that the real axis in the Z-plane is mapped into the negative real axis in the w-plane by $w = Z^2$.

Whenever $u = u_o$ is a constant greater than zero, then the equilateral hyperbola $u_o = x^2 - y^2$ is mapped into the line u u_o under the mapping $w = Z^2$ (see figure 13).

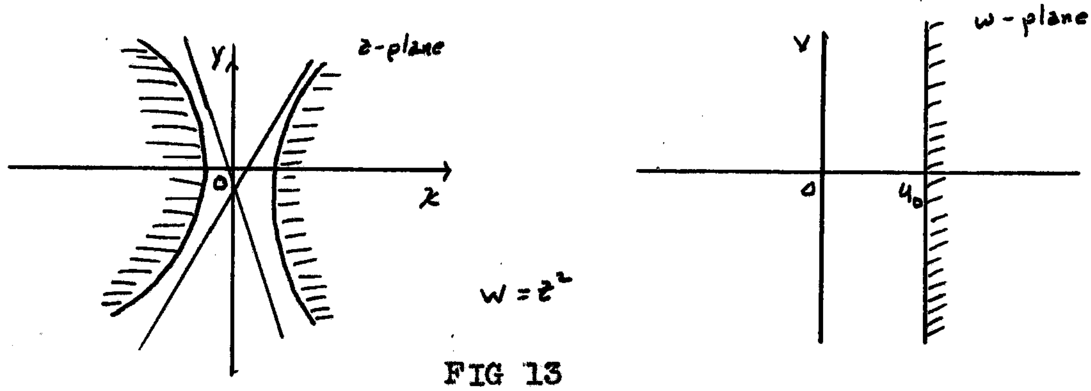


FIG 13

Likewise if $v = v_0$ (a constant) then the equilateral hyperbola $v_0 = 2xy$ is mapped into the line $v = v_0$ under the mapping $w = Z^2$.

When n is a positive integer, the transformation

$$w = Z^n, \text{ or } \quad \rho e^{i\phi} = r^n e^{in\theta},$$

maps the angular region $r \geq 0$, $0 \leq \theta \leq \pi/n$, onto the upper half of the w-plane(figure 14), since $\rho = r^n$ and $\phi = n\theta$. It transforms a circular arc

$$r = r_0 \ (\theta_0 \leq \theta \leq \theta_0 + 2\ /n)$$

into the circle $\rho = r_0^n$. Both mappings are one to one.

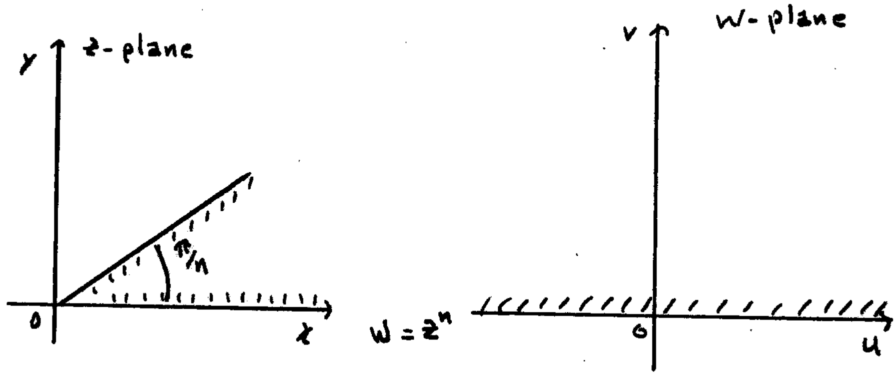


FIG. 14

14. The Function $w = \log Z$

In our consideration of the transformation $w = \log Z$, we restrict ourselves to the principal value of log Z, that is

$$-\pi \leq \arg Z \leq \pi .$$

We see then, that $dw/dZ \quad 1/Z$, so that the mapping $w = \log Z$ is conformal for all $Z \neq 0$. Now observe, that

$$w = \log Z = \log |r| e^{i\theta}, \quad Z = |r| e^{i\theta}$$

then $w = \log r + i\theta$. But $w \quad u + iv$. Therefore $u + iv = \log r + i\theta$ implies that

$$u = \log r \text{ and } v = \theta.$$

Suppose $\theta = 0$, then $v = 0$ and the mapping $w = \log Z$ takes the positive real axis in the Z-plane into the real axis in the w-plane.

Now suppose $\theta = \alpha$, then $v = \alpha$ and $w = \log r + i\alpha$.

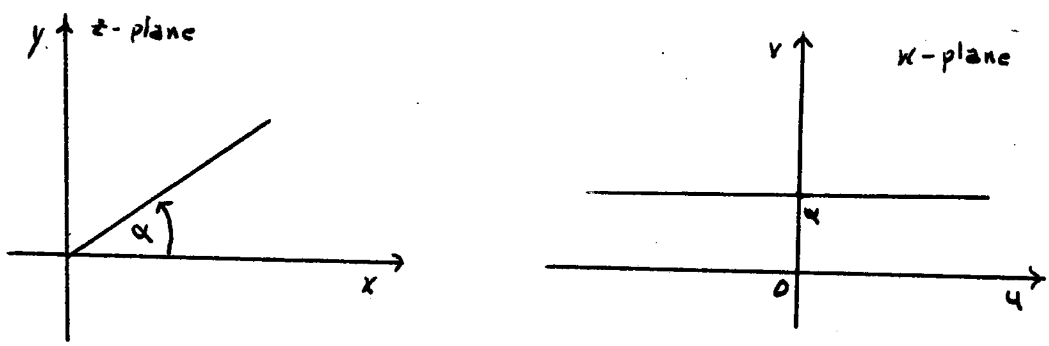
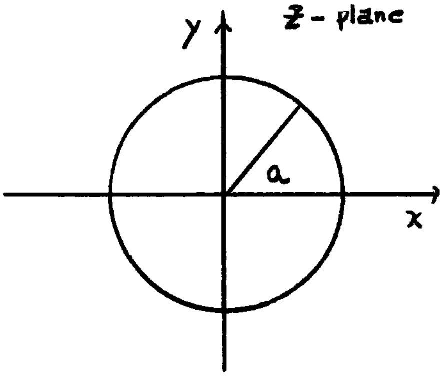


FIG. 15

Hence the mapping $w = \log Z$ maps the ray $\theta = \alpha$ into the line parallel to the u-axis, α units above the u-axis (figure 15).

Example.- Find the image of the circle $|Z| = a$, where $a > 0$, under the mapping $w = \log Z$.

Solution: Observe that, $w = \log a + i\theta$. Hence the circle $|Z| = a$, where $a > 0$ goes into the segment $w = \log a + i\theta$, where $-\pi < \theta < \pi$ (see figure 16).



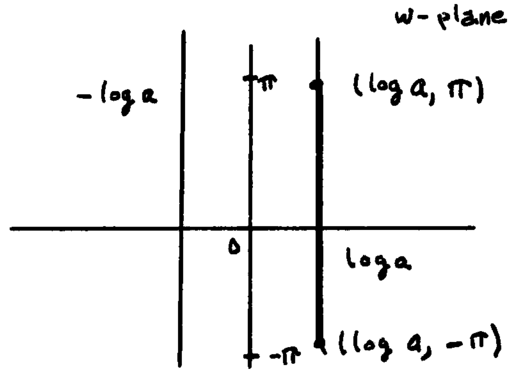


FIG. 16

15. The Inverse Transformation

The transformation

$$w = 1/Z \text{ or } Z = 1/w$$

sets up a one to one correspondence between points in the Z-plane and points in the w-plane, except for the points $Z = 0$ and $w = 0$, which have no image. This mapping is conformal except at $Z = 0$ and $w = 0$ since $dw/(dZ) = -1/Z^2$.

In polar coordinates the transformation becomes

$$\rho e^{i\phi} = \frac{1}{r} e^{-i\theta}$$

where $Z = |Z| e^{-i\theta} = r e^{i\theta}$, and $w = \rho e^{i\phi}$.

When cartesian coordinates are used, the equation

$$w = u + iv = 1/(x + iy)$$

gives the relations

$$u = x/(x^2 + y^2), \quad v = -y/(x^2 + y^2)$$

and

$$x = u/(u^2 + v^2), \quad y = -v/(u^2 + v^2).$$

Example.- Find the image of the straight line

$$(1) \quad y = mx + b$$

under the transformation $w = 1/Z$.

Solution: Recall

$$(2)\ x = (Z - Z)/2 \text{ and } y = (Z - Z)/2i.$$

Using 2 we see that (1) becomes

$$(3)\ (Z - Z) = im(Z + Z) + 2ib, \text{ or}$$

$$(1 - im)Z - (1 + im)Z - 2ib = 0,$$

that is

$$(4)\ (im - 1)Z + (im + 1)Z + 2ib = 0.$$

Now under the transformation $w = 1/Z$, equation (4) becomes

$$(5)\ 2ibw\bar{w} + (im + 1)w + (im - 1)\bar{w} = 0.$$

Set $w \quad u + iv$ and $\bar{w} = u - iv$, the $w\bar{w} = u + v$. Thus (5) becomes

$$(6)\ 2ib(u^2 + v^2) + (im + 1)(u + iv) + (im - 1)$$

$$\text{times } (u - iv) = 0,$$

which becomes

$$2ib(u + v^2) + 2i(mu + v) = 0,$$

that is

$$(7)\ b(u^2 + v^2) + mu + v = 0.$$

Observe that, (7) is a circle passing through the origin if $b \neq 0$ and it is a straight line passing through the origin if $b = 0$.

Assume $b \neq 0$, then

$$b(u^2 + v^2) + mu + v = u^2 + v^2 + mu/b + v/b$$

$$u^2 + mu/b + v^2 + v/b = 0,$$

which upon completing the square becomes

$$b\left[(u^2 + mu/b + m^2/(4b^2)) + (v^2 + r/b + 1/(4b^2))\right] -$$

$$m^2/(4b^2) - 1/(4b^2) = 0.$$

Hence equation (7) becomes

$$(8)\ (u + m/(2b))^2 + (v + 1/2b)^2 = (m^2 + 1)/(4b^2).$$

Thus the center of the circle (7) is

$$(-m/(2b), -1/(2b))$$

and its radius is

$$m^2 + 1/(2b).$$

Thus we see that the transformation $w = 1/Z$ takes the straight line $y = mx + b$ into the circle $b(u^2 + v^2) + mu + v = 0$, if $b \neq 0$, and into the straight line $mu + v = 0$ if $b = 0$.

Exercise.- Find the image of the circle

$$(1)\ (x^2 + y^2) + \quad x + by + c = 0$$

under the transformation $w = 1/Z$.

Hint: Set $x = (Z + Z)/2$ and $y = (Z - Z)/2$. Then (1) becomes

$$(Z^2 + Z^2 + 2Z\bar{Z} - Z^2 - Z^2 + 2Z\bar{Z}) + 2\ (Z + Z) +$$
$$2bi(Z - Z) + 4c = 0,$$

which becomes

$$(2)\ 2Z\bar{Z} + (a - bi)Z + (\ + bi)Z + 2c = 0.$$

Now under the transformation $w = 1/Z$, (2) becomes

$$(3)\ 2cw\bar{w} + (a + bi)w + (a - bi)w + 2.$$

Note that (3) is a straight line through the origin if $c = 0$, and is a circle not passing through the origin if $c \neq 0$.

Put (3) in the standard form and find the center and radius for the case where $c \neq 0$.

16. Inversion Transformation With Respect to a Given Circle

Consider the circle $C: |Z| = R$. Let Z be any point exterior

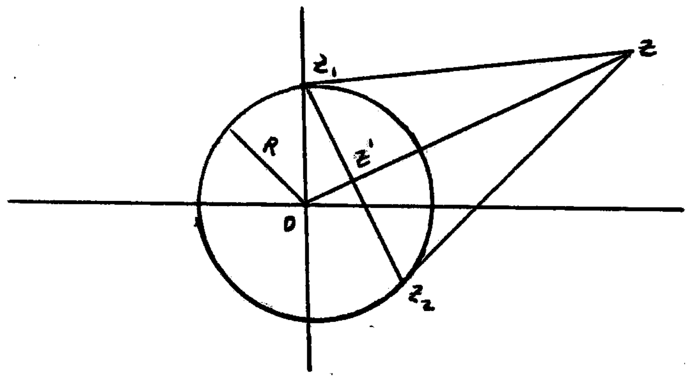


FIG. 17

to C. Draw from Z, tangents t C, at Z_1 and Z_2. Then connect Z_1 to Z_2; the point Z', of the intersection Z_1, Z_2 and ZO is the inverse point to Z and points Z and Z' are called conjugate points. Every point lying on the circunference of C is its own conjugate.

We claim that

$$|Z|\,|Z'| = R^2,$$

where Z and Z' are conjugate points relative to circle C.

Proof: Triangle Z_1OZ' is similar to triangle ZOZ_1, since their corresponding angles are equal, that is,

angle $OZ'Z_1$ equals angle $Z'Z_1Z$,

angle OZ_1Z' equals angle $Z'ZZ_1$, and

angle Z_1OZ' belongs to both the triangles.

Hence

$$(OZ_1)/\,(OZ) = (OZ')\;/\;(OZ_1),$$

that is $|Z|\,|Z'| = |Z_1|^2 = R^2$. Therefore $|Z|\,|Z'| = R^2$. If $R = 1$, then $|Z'| = 1/Z$, since $|Z|\,|Z'| = 1$ and $|Z'| = 1/Z$, if $R = 1$.

Thus the inversion transformation is conformal, since arg Z = arg Z'.

In the inverse transformation $w = 1/Z$ the $\arg w = -\arg Z$ and $|w| = 1/Z$. The inversion transformation maps every point inside a circle in the Z-plane outside a specific circle in the w-plane and maps every point outside a given circle in the Z-plane inside a specific circle in the w-plane.

The point $w = 0$ is not mapped into any point in the finite Z-plane. However if we make the radius of the circle in the Z-plane sufficiently large, the images of all points Z outside the large circle are made to fall within an arbitrarily small neighborhood of the point $w = 0$.

Formally, the point $Z = \infty$ is the image of the point $w = 0$ under the transformation $w = 1/Z$. That is, whenever a statement is made about the behavior of a function at $Z = \infty$, we mean precisely the behavior of the function at $Z' = 0$, where Z' is $1/Z$.

17. Bilinear Transformations

The transformation

$$W = \frac{\alpha Z + \beta}{\gamma Z + \delta}, \quad (\alpha\delta - \beta\gamma \neq 0),$$

where α, β, γ and are complex constants, is called the linear fractional transformation or the bilinear transformation. We abbreviate it $w = T(Z)$. Observe that

$$\frac{dw}{dZ} = \frac{(\gamma Z + \delta) - (\alpha Z + \beta)\gamma}{(\gamma Z + \delta)^2} = \frac{\alpha\delta - \beta\gamma}{(\gamma Z + \delta)^2}.$$

If $\alpha\delta - \beta\gamma = 0$, then $dw/dZ = 0$ for all Z in the complex plane. Hence

$$\frac{\alpha z + \beta}{\gamma z + \delta} = C,$$

Where C is some complex constant. Therefore $\alpha Z + \beta = C(\gamma Z + \delta)$, that is

$$\frac{\alpha z + \beta}{\gamma z + \delta},$$

is either a constant or meaningless if $\alpha\delta - \beta\gamma = 0$.

By considering

$$W = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \alpha\delta - \beta\gamma \neq 0,$$

we easily see that we can let $Z = -\delta/\gamma$ correspond to $w \longrightarrow \infty$.

We claim that

$$W = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \alpha\delta - \beta\gamma \neq 0$$

is a one to one mapping.

Proof: Suppose there exist distinct points Z_0 and Z_1 such that $w_0 = w_1$, that is

$$\frac{\alpha z_0 + \beta}{\gamma z_0 + \delta} = \frac{\alpha z_1 + \beta}{\gamma z_1 + \delta} \Longrightarrow (\alpha z_0 + \beta)(\gamma z_1 + \delta) = (\alpha z_1 + \beta)(\gamma z_0 + \delta),$$

$$\Longrightarrow \alpha z_0 \gamma z_1 + \alpha z_0 \delta + \beta\gamma z_1 + \beta\delta = \alpha z_1 \gamma z_0 + \alpha z_1 \delta + \beta\gamma z_0 + \beta\delta,$$

$$\Longrightarrow \beta\gamma z_1 + \alpha z_0 \delta = \beta\gamma z_0 + \alpha z_1 \delta$$

$$\Longrightarrow (\beta\gamma - \alpha\delta) z_1 = (\beta\gamma - \alpha\delta) z_0 \Longrightarrow z_1 = z_0 .$$

Thus we have a contradiction since we assumed Z_0 and Z_1 to be distinct points. Therefore

$$w = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \alpha\delta - \beta\gamma \neq 0$$

is a one to one mapping.

The bilinear transformation always transforms circles into circles and lines into lines.

Solving the equation

$$w = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \alpha\delta - \beta\gamma \neq 0$$

for Z in terms of w, we have

$$w = \frac{\alpha z + \beta}{\gamma z + \delta} \Longrightarrow w\gamma z + w\delta = \alpha z + \beta$$

$$\Longrightarrow (\gamma w - \alpha) z = -\delta w + \beta$$

$$\Longrightarrow (1)\ z = \frac{-\delta w + \beta}{\gamma w - \alpha} = \frac{(-\delta) w + \beta}{\gamma w - \alpha},$$

which is the same form as

$$w = \frac{\alpha z + \beta}{(\gamma z + \delta)}.$$

In one we see that we may let $w = \alpha/\gamma$ corresponds to $Z = \infty$.

We want to show now that the bilinear transformation is a product of inversions, translations and rotations.

We introduce the following notations:

Let (i) $T_a Z = Z + a$, where a is a complex number.

(ii) $S_b Z = bZ$, where b is a complex number.

(iii) $V_Z = 1/Z$.

Proof: (i) Suppose $\gamma \neq 0$. Then

$$W=\frac{\alpha z+\beta}{\gamma z+\delta}=\frac{1}{\gamma}\left(\frac{\alpha z+\beta}{z+\frac{\delta}{\gamma}}\right)=\frac{1}{\gamma}\left[\frac{\alpha(z+\frac{\delta}{\gamma})-\alpha\frac{\delta}{\gamma}\beta}{z+\frac{\delta}{\gamma}}\right]$$

$$=\frac{1}{\gamma}\left[\alpha-\frac{\alpha\delta-\beta\gamma}{\gamma^2(z+\delta/\gamma)}\right].$$

Set $\Delta=\alpha\delta-\beta\gamma$. Now observe that,

$$T_{\frac{\delta}{\gamma}}Z=Z+\frac{\delta}{\gamma}$$

and

$$S_\gamma T_{\frac{\delta}{\gamma}}Z=S_\gamma\left(Z+\frac{\delta}{\gamma}\right)=\gamma Z+\delta,$$

upon applying an inversion to the above we have

$$VS_\gamma T_{\frac{\delta}{\gamma}}Z=V(\gamma Z+\delta)=\frac{1}{\gamma z+\delta}.$$

Now upon applying the rotation $S_{-\frac{\Delta}{\gamma}}$ and then the translation $T_{\frac{\delta}{\gamma}}$ to the above we have

$$S_{-\frac{\Delta}{\gamma}}VS_\gamma T_{\frac{\delta}{\gamma}}z=S_{-\frac{\Delta}{\gamma}}\left[\frac{1}{\gamma z+\delta}\right]=\frac{-\Delta}{\gamma(\gamma z+\delta)},$$

and

$$T_{\frac{\delta}{\gamma}}S_{-\frac{\Delta}{\gamma}}VS_\gamma T_{\frac{\delta}{\gamma}}z=T_{\frac{\delta}{\gamma}}\left[\frac{-\Delta}{\gamma(\gamma z+\delta)}\right]=\frac{\alpha}{\gamma}-\frac{\Delta}{\gamma(\gamma z+\delta)},$$

but since $\Delta=\alpha\delta-\beta\gamma$, we have

$$T_{\frac{\delta}{\gamma}}S_{-\frac{\Delta}{\gamma}}VS_\gamma T_{\frac{\delta}{\gamma}}z=\frac{\alpha}{\gamma}-\frac{\alpha\delta-\beta\gamma}{\gamma(\gamma z+\delta)}=\frac{\alpha z+\beta}{\gamma z+\delta}=w.$$

(ii) Now suppose $\gamma=0$. Then since $\alpha\delta-\beta\gamma\neq 0$, we must have $\alpha\neq 0$ and $\delta\neq 0$, so that the following situation exists:

$$W=\frac{\alpha z+\beta}{\delta}=\frac{\alpha}{\delta}z+\frac{\beta}{\delta}.$$

Hence

$$\omega = \frac{\alpha z + \beta}{\delta} = \frac{\alpha}{\delta} z + \frac{\beta}{\delta} = V_{\frac{1}{\delta}} T_\beta S_\alpha z.$$

We may therefore conclude that the bilinear transformation is a product of inversions, translations and rotations. Q.E.D.

The bilinear transformation is the most general mapping that takes circles into circles. We now define cross-ratio.

Definition.- Let Z_1, Z_2, Z_3, and Z_4 be four distinct points in the Z-plane, then any expression of the form

$$\frac{z_1 - z_4}{z_1 - z_2} \cdot \frac{z_3 - z_2}{z_3 - z_4},$$

is called the cross-ratio of the points Z_1, Z_2, Z_3 and Z_4.

Theorem.-The cross-ratio is invariant under the bilinear transformation.

Proof: Let Z_1, Z_2, Z_3 and Z_4 be any distinct points in the Z-plane and let w_1, w_2, w_3 and w_4 be their correlates in the w-plane.

We want to that

$$(1)\quad \frac{z_1 - z_4}{z_1 - z_2} \cdot \frac{z_3 - z_2}{z_3 - z_4} = \frac{W_1 - W_4}{W_1 - W_2} \cdot \frac{W_3 - W_2}{W_3 - W_4}$$

Recall that

$$W_i = \frac{\alpha z_i + \beta}{\gamma z_i + \delta}, \qquad \text{where } i = 1, 2, 3, \text{ and } 4.$$

Observe that

$$(2)\quad W_i - \omega_j = \frac{\alpha z_i + \beta}{\gamma z_i + \delta} - \frac{\alpha z_j + \beta}{\gamma z_j + \delta},$$

$i \neq j$ and, i and $j = 1, 2, 3,$ and 4. Then

$$w_i - w_j = \frac{(\alpha z_i + \beta)(\gamma z_j + \delta) - (\alpha z_j + \beta)(\gamma z_i + \delta)}{(\gamma z_i + \delta)(\gamma z_j + \delta)}$$

$$= \frac{\alpha\delta(z_i - z_j) + \beta\gamma(z_j - z_i)}{(\gamma z_i + \delta)(\gamma z_j + \delta)}$$

$$= \frac{(z_i - z_j)(\alpha\delta - \beta\gamma)}{(\gamma z_i + \delta)(\gamma z_j + \delta)}.$$

Now we have

$$\frac{w_1 - w_4}{w_1 - w_2} \cdot \frac{w_3 - w_2}{w_3 - w_4} = \frac{\dfrac{(z_1 - z_4)(\alpha\delta - \beta\gamma)}{(\gamma z_1 + \delta)(\gamma z_4 + \delta)} \cdot \dfrac{(z_3 - z_2)(\alpha\delta - \beta\gamma)}{(\gamma z_3 + \delta)(\gamma z_2 + \delta)}}{\dfrac{(z_1 - z_2)(\alpha\delta - \beta\gamma)}{(\gamma z_1 + \delta)(\gamma z_2 + \delta)} \cdot \dfrac{(z_3 - z_4)(\alpha\delta - \beta\gamma)}{(\gamma z_3 + \delta)(\gamma z_4 + \delta)}}$$

$$= \frac{z_1 - z_4}{z_1 - z_2} \cdot \frac{z_3 - z_2}{z_3 - z_4}.$$

Therefore our assertion. Q.E.D.

Note that under the bilinear mapping three distinct points are independent, that is, three distinct points determin the bilinear mapping. Thus if we are given any three distinct points in the Z-plane, say Z_1, Z_2, and Z_3, there exist a bilinear map taking $Z_1 \longrightarrow w_1$, $Z_2 \longrightarrow w_2$, and $Z_3 \longrightarrow w_3$.

Example.- Consider the points $Z = 1, 0,$ and ∞. Find a bilinear mapping which takes $1 \longrightarrow 0$, $0 \longrightarrow i$, and $\infty \longrightarrow 1$.

Solution: In

$$\frac{z_1 - z_4}{z_1 - z_2} \cdot \frac{z_3 - z_2}{z_3 - z_4} = \frac{w_1 - w_4}{w_1 - w_2} \cdot \frac{w_3 - w_2}{w_3 - w_4}$$

Let $Z_4 = Z$ and $w_4 = w$. Then observe that with the proper interpertation the following is true:

$$\frac{0-w}{0-i}\cdot\frac{1-i}{1-w}=\frac{1-z}{1-0}\cdot\frac{\infty-0}{\infty-z}$$

$$\Longrightarrow \frac{w}{i}\left(\frac{i-1}{w-1}\right)=1-z\text{, where } \frac{\infty-0}{\infty-z} \text{ goes to 1.}$$

which implies that $w(i - 1) = wi(1 - Z) - i(1 - Z)$.

Hence

$$w=\frac{-i(1-z)}{zi-1} \quad\text{or}\quad w=\frac{z-1}{z+i}$$

Note that if (i) $Z = 1$, then $w = 0/(1 + i) = 0$,

(ii) $Z = 0$, then $w = -1/i = -i/-1 = i$,

(iii) $Z = \infty$, then $w = (\infty - 1)/(\infty + i) = 1$.

Exercise.- Find the mobius (bilinear) transformation which takes the set of points (a, b, c) in the Z-plane into the set of points $(0, 1, \infty)$ in the w-plane.

18. Some Special Conformal Transformations

(i) Take the unit circle into the unit circle. Note that inverse points under a bilinear transformation go into inverse points. The inverse of $(\bar{a})$ is such $a\cdot\bar{a} = 1$, that is

$$a \longrightarrow 0$$
$$1/a \longrightarrow \infty.$$

We want a bilinear transformation which takes $a \longrightarrow 0$ and $1/a \longrightarrow \infty$ (see figure 18).

The transformation

$$w = k\,\frac{z-a}{z-\frac{1}{\bar{a}}} = k\bar{a}\,\frac{z-a}{\bar{a}z-1}$$

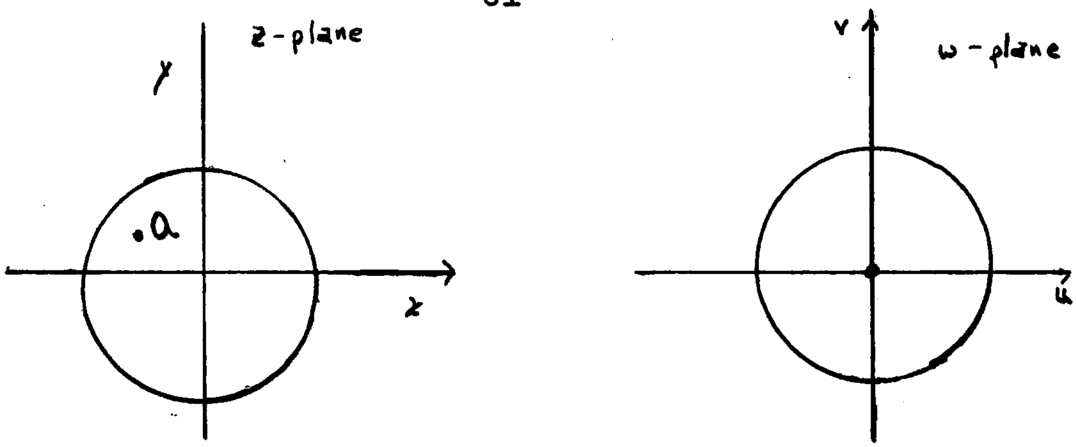


FIG. 18

is such a mapping but we want to determine the constant K.

Observe that

$$|w| = |K| \left| \frac{z-a}{z-\frac{1}{\bar{a}}} \right| = |K\bar{a}| \left| \frac{z-a}{\bar{a}z-1} \right|$$

for Z = 1 we must have

$$|w| = 1 = |K\bar{a}| \left| \frac{1-a}{\bar{a}-1} \right| = |K\bar{a}|.$$

Hence $K\bar{a} = e^{i\alpha}$. Therefore

$$w = e^{i\alpha} \frac{z-a}{\bar{a}z-1}$$

is the general bilinear transformation which takes the unit circle $|Z| \leq 1$ into the unit circle $|w| \leq 1$.

(ii) Find the bilinear transformation which takes the upper half- plane (z-plane) into the unit circle (in the w-plane). See figure 19.

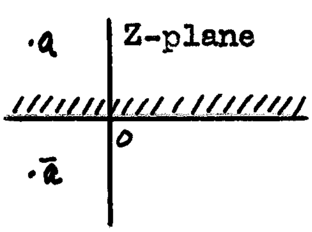
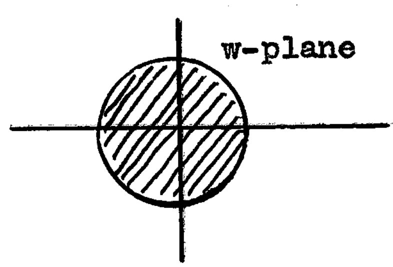


FIG. 19



Since the upper half-plane may be considered as a circle with an infinite radius, the inverse of   is the complex conjugate of . Hence we want a mapping taking

$$a \longrightarrow 0$$

$$\bar{a} \longrightarrow \infty.$$

The mapping which does the above is

$$w = K\,\frac{z-a}{z-\bar{a}},$$

then

$$|w| = |K|\left|\frac{z-a}{z-\bar{a}}\right|.$$

For Z = 0, $1 = |w| = |K|\left|\frac{-a}{-\bar{a}}\right| = |K|$.

Hence $K = e^{i\lambda}$ and $w = e^{i\lambda}\,z-a/z-\bar{a}$ is the mapping which takes imaginary Z $\geq$ 0 into $|Z| \leq 1$.

<u>Example</u>.- Find a conformal mapping of the region in figure 20, into the unit circle.

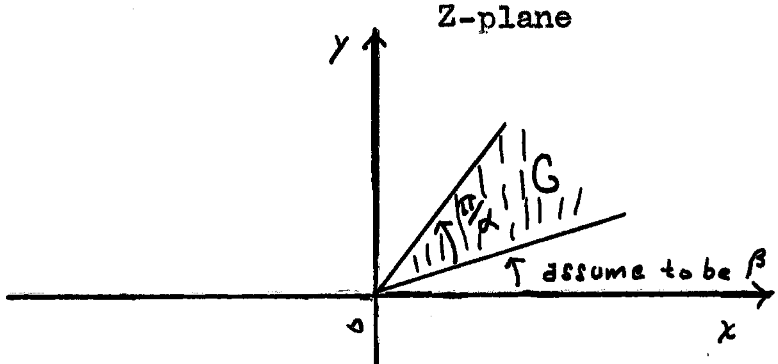


FIG. 20

(i) $\sigma = e^{i(-\beta)}z = e^{-i\beta}z$ is a rotation of all points belonging to the region G of figure 20 through an angle of in the z-plane (see figure 21).

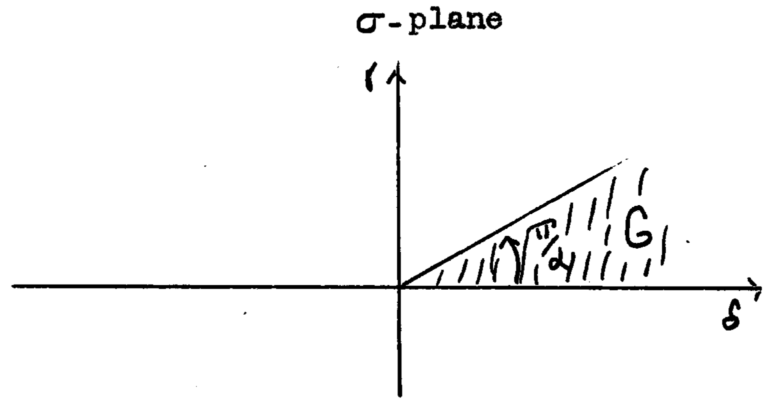


FIG. 21

(ii) $\zeta = \sigma^{\alpha} = e^{-i\alpha\beta} z^{\alpha}$, is a magnification of G into the upper half-plane (See figure 22).

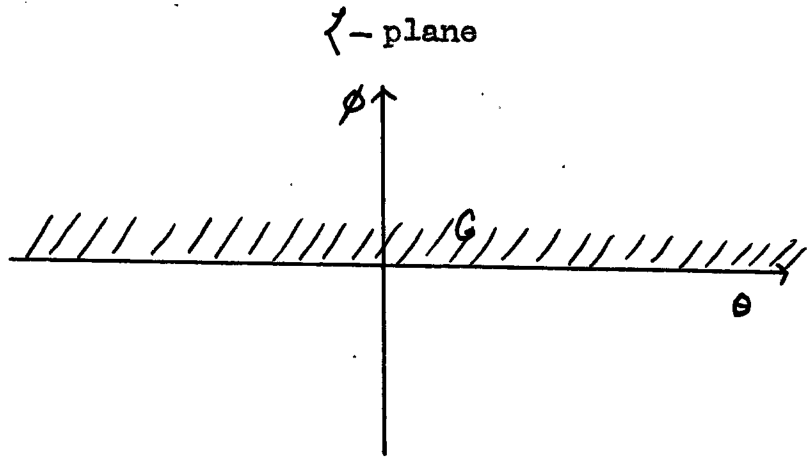


FIG. 22

(iii) $$\omega = e^{i\lambda}\frac{\zeta - a}{\zeta - \bar{a}} = e^{i\lambda}\frac{(e^{-i\alpha\beta} z^{\alpha} - a)}{(e^{-i\alpha\beta} z^{\alpha} - \bar{a})}$$

is the desired transformation of the region G of the Z-plane into the unit circle in the w-plane.

<u>Exercise</u>.- Find the bilinear transformation which takes the region G in (figure 23) the Z-plane into the unit circle

in the w-plane.

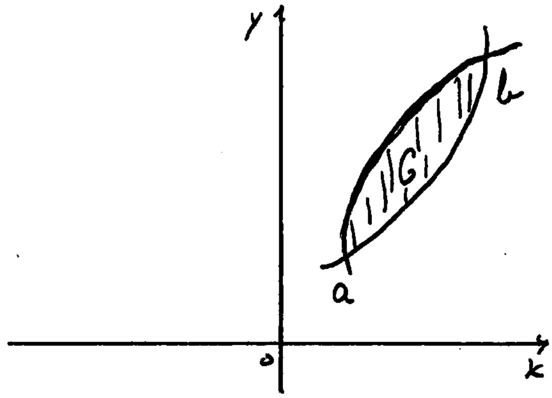


FIG. 23

19. Inverse Points With Respect to a Circle

Theorem.- If p and q are inverse points with respect to the circle C: $|Z - Z| = \rho$, then $|p - Z_0|\,|q - Z_0| = \rho^2$.

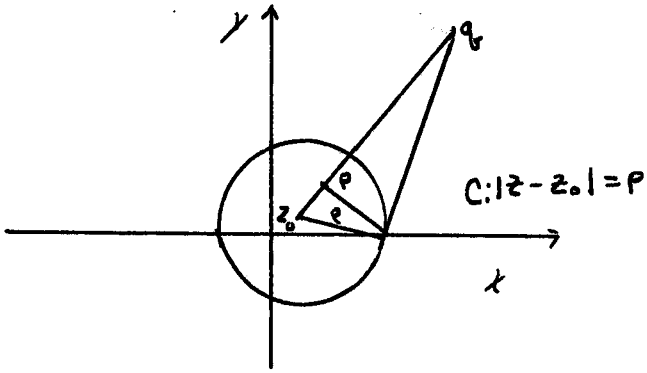


FIG. 24

Proof: Our hypothesis implies that

$$p = Z_0 + \ell e^{i\lambda}$$

and

$$q = Z_0 \frac{\rho^2}{\ell} e^{i\lambda}$$

Let Z be any point on C, then consider

but $Z = Z_0 + \rho e^{i\theta}$.

Thus we see that

$$\left|\frac{z-p}{z-q}\right| = \left|\frac{\rho e^{i\theta} - l e^{i\lambda}}{\rho e^{i\theta} - \frac{\rho^2}{l} e^{i\lambda}}\right| = \frac{l}{\rho}\left|\frac{\rho e^{i\theta} - l e^{i\lambda}}{l e^{i\theta} - \rho e^{i\lambda}}\right| = \frac{l}{\rho}$$

Therefore

$$\left|\frac{z-p}{z-q}\right| = \frac{l}{\rho}$$

is another form of the equation of the circle of inversion with respect to which p and q are inverse points. Q.E.D.

If $\left|\frac{z-p}{z-q}\right| = k$

is the equation of a circle, then the bilinear transformation takes it into a circle and its inverse points into inverse points.

Proof: Consider the equation $\left|\frac{z-p}{z-q}\right| = k$

Recall that the transformation $w = \frac{\alpha z + \beta}{\gamma z - \delta}$

maps straight lines into straight lines and circles into circles into circles. Thus C: $|Z - Z_0| = e$ is mapped into its image circle in the w-plane by

$$(1) \quad W = \frac{\alpha z + \beta}{\gamma z + \delta}$$

We want to show that under the bilinear mapping (1), points p and q go into a pair of inverse points p' and q'. Observe that solving (1) for Z we obtain

$$z = \frac{-\delta w + \beta}{\gamma w - \alpha}$$

Thus

$$\left|\frac{z-p}{z-q}\right| = k$$

goes into

$$\left|\frac{\dfrac{-\delta w+\beta}{\gamma w-\alpha} - p}{\dfrac{-\delta w+\beta}{\gamma w-a} - q}\right| = \left|\frac{\dfrac{-\delta w+\beta-\gamma w p+\alpha p}{\gamma w-\alpha}}{\dfrac{-\delta w+\beta-\gamma w q+\alpha q}{\gamma w-\alpha}}\right|$$

$$= \left|\frac{w-\dfrac{\alpha p+\beta}{\delta+\gamma p}}{w-\dfrac{\alpha q+\beta}{\delta+\gamma q}}\right| \cdot \left|\frac{(\delta+\gamma p)}{(\delta+\gamma q)}\right| = k$$

Thus

$$(2) \qquad \left|\frac{w-\dfrac{\alpha p+\beta}{\delta+\gamma p}}{w-\dfrac{\alpha q+\beta}{\delta+\gamma q}}\right| = \frac{k}{\left|\dfrac{(\delta+\gamma p)}{(\delta+\gamma q)}\right|} = L$$

Therefore, since (2) is a form of the equation of the circle with respect to p' and q' as inverse points (L is a complex constant), we may conclude that the our assertion holds.

20. <u>The Function $w = Z^{\frac{1}{2}}$ and $w = e^{z}$</u>

The multiple-valued function $w = Z^{\frac{1}{2}} = \sqrt{r}e^{i\frac{\theta}{2}}$, where $Z = re^{i\theta}$, takes on two values at each point Z except the origin, depending on the choice of θ. One value is the negative of the other because $e^{(\frac{i\theta}{2})}$ changes in sign alone when θ is increased by 2π.

Set $Z = x + iy$ and $w = u + iv$. Then $w^2 = Z$ implies that $u^2 - v^2 + 2iuv = x + iy$, which implies that (1) $x = u^2 - v^2$ and (2) $y = 2uv$. Suppose we were to square y. Then $y^2 = 4u^2v^2$. But from (1) we have that

$$v^2 = u^2 - x, \text{ so that}$$
$$y^2 = 4u^2(u^2 - x)$$

If $u = 0$, then from (1) $x = -v^2$ and (2) becomes $y = 0$. Then the nonepositive real axis in the Z-plane goes into the imaginary axis in the w-plane under the mapping $w = Z^{\frac{1}{2}}$.

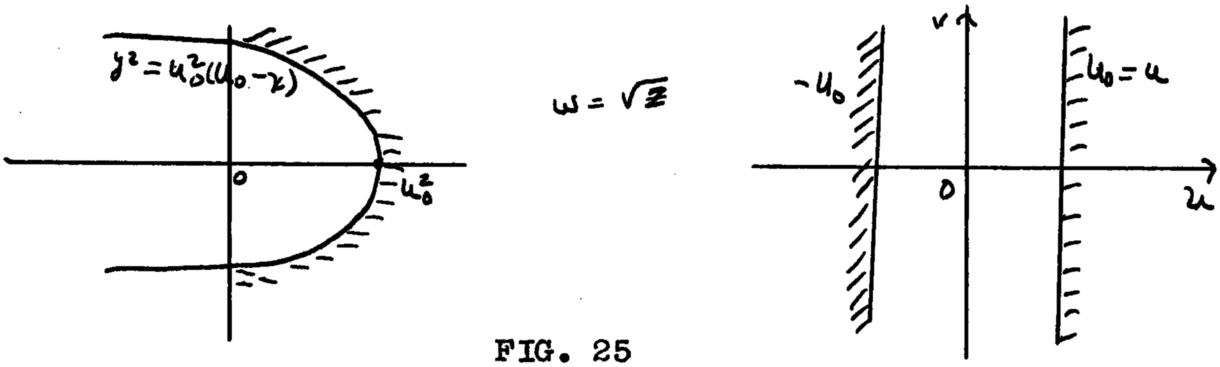


FIG. 25

If $u = u_0$ (a constant), then the transformation

$$w = Z^{\frac{1}{2}}$$

takes all the points belonging to the region outside the parabola $y^2 = u_0^2(u_0^2 - x)$ into the real part of w greater than or equal to u_0 if $u_0 < 0$ and into the real part of w less than or equal to u_0 if u_0 is negative (see figure 25).

Now we consider the transformation $w = e^z$, or

$$\rho e^{i\phi} = e^x e^{iy},$$

where $Z = x + iy$. Thus $w = \rho e^{i\theta}$, can be written $\rho = e^x$, $\phi = y$.

Suppose $y = y_0$ (a constant). Then we see that under the transformation $w = e^z$, the line $y = y_0$ goes into the ray

$$w = e^z$$

Now suppose we fix x, that is $x = x_0$ (a constant), then

$$\rho_0 = e^{x_0}$$

Thus we see that the line $x = x_0$ goes into the circle

$$|w| = \rho_0$$

see figure 26.

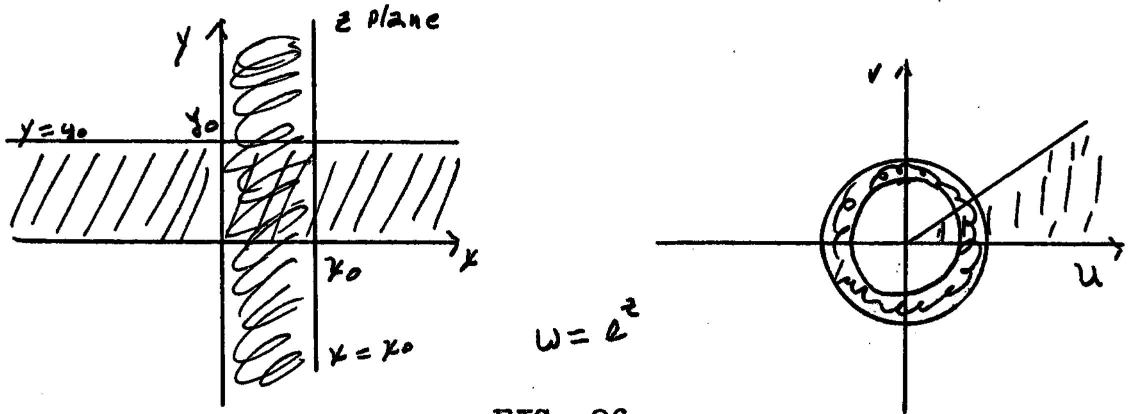


FIG. 26

We now consider some transformations of particular regions by $w = e^z$.

(i) The strip $-\infty < x < +\infty$, $0 \leq y < \pi$, is transformed by $w = e^z$ into the imaginary axis greater than or equal to zero, that is the upper half-plane minus the negative real axis.

(ii) The strip $-\infty < x \leq 0$, $0 \leq y < 1$, is transformed by $w = e^z$ into $|w| \leq 1$, where $0 \leq \arg w < \pi$.

21. <u>The function $w = Z + 1/Z$</u>

Set $Z = re^{i\theta}$, then $1/Z = 1/(r)e^{-i\theta}$, and $w = r + 1/(r)e^{-i\theta}$

$$= r\cos\theta + i\sin\theta + \frac{1}{r}\cos - \frac{i}{r}\sin\theta$$

$$= \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

But $w = u + iv$, hence

(1) $u = (r + 1/r)\cos\theta$, and

(2) $v = (r - 1/r)\sin\theta$.

Suppose $r = 1$. Then $w = 2\cos\theta$. Thus we see that

$$w = 2\cos\theta$$

traverses the interval between -2 and +2 twice as θ ranges over the interval $[0, 2\pi]$, (see figure 27)

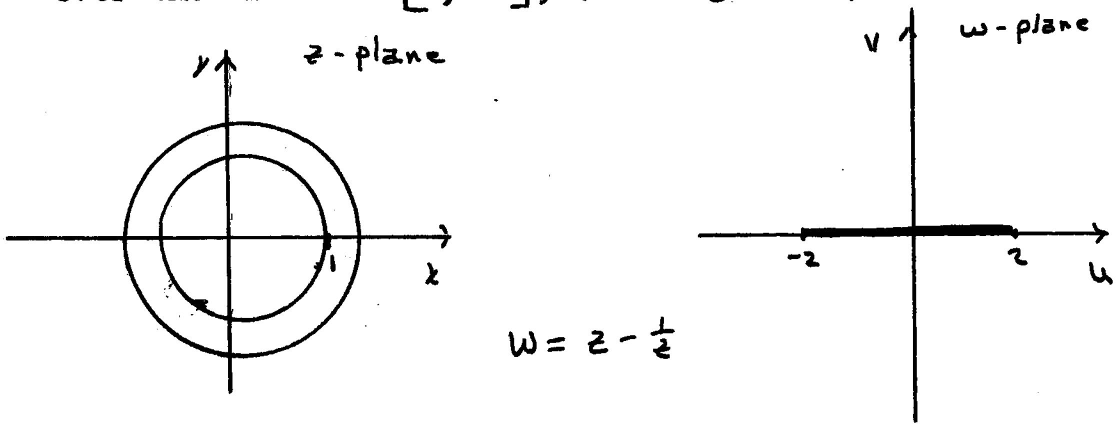


FIG. 27

Suppose $r > 1$, then

$$(1)\quad \frac{u^2}{(r+\frac{1}{r})^2} + \frac{v^2}{(r-\frac{1}{r})^2} = 1,$$

which is the equation of an ellips in the w-plane. Now observe

$$c^2 = a^2 - b^2 \text{ (where } c^2 \text{ are the foci)}$$

$$(r + 1/r)^2 - (r - 1/r)^2 = 4,$$

where $a = (r + 1/r)$ and $b = (r - 1/r)$, we have that $c = \pm 2$, so that the foci of the ellipse are at -2 and +2. This ellipse goes into the interval $-2 \leq u \leq +2$, as r approaches 1.

If $r < 1$ we get the same set of ellipses as in the case where $r > 1$. The limit of the interior of the unit circle goes into the interval $-2 \leq u \leq +2$, and the limit of the exterior of the unit circle goes into the same region. Thus the region inside the unit circle the whole w-plane exclusive of the interval $w = 2 \cos \theta$, $0 \leq \theta \leq 2\pi$.

22 <u>Some Special Examples</u>

(i) If $w = \cosh Z$, prove that the area of the region of the w-plane which corresponds to the rectangle bounded by the lines $x = 0$, $x = 2$, $y = 0$ and $y = 1/4\pi$ is

$$\frac{\pi \sinh 4 - 8}{16}$$

Proof: Let Δ represente the rectangle in the Z-plane bounded by the sides $x = 0$, $x = 2$, $y = 0$, and $y = 1/4\pi$. Let D be the closed domain of the w-plane which corresponds to Δ. Since $f'(Z) \neq 0$ and Δ and D are closed we have that

$$(1) \quad A = \iint_D du\,dv = \iint_\Delta |f'(z)|^2\,dx\,dy$$

where A is the area of D. Observe $f'(Z) = u_x + iv_x$ and

$$|f'(Z)|^2 = u_x^2 + v_x^2.$$

Thus (1) becomes

$$\int_0^{\pi/4}\int_0^2 \left[u_x^2 - V_x^2\right] dx\,dy$$

Now observe

$$w = \cosh Z = \cosh(x + iy)$$
$$= \cosh x \cosh iy + \sinh x \sinh iy.$$

Now recall,

$$\cosh r = \frac{e^r + e^{-r}}{2}, \text{ then } \cosh iy = \frac{e^{iy} + e^{-iy}}{2}$$

and

$$\sinh r = \frac{e^r - e^{-r}}{2}, \text{ then } \sinh iy = \frac{e^{iy} - e^{-iy}}{2}$$

Thus

$$w = \cosh x \cos y + i \sinh x \sin y.$$

Hence

$$u = \cosh x \cos y, \quad v = \sinh x \sin y,$$

and

$$u_x = \sinh x \cos y, \quad v_x = \cosh x \sin y.$$

Thus we have

$$
\begin{aligned}
u_x^2 + v_x^2 &= \sinh^2 x \cos^2 y + \cosh^2 x \sin^2 y \\
&= \sinh^2 x - \sinh^2 x \sin^2 y + \cosh^2 x \sin^2 y \\
&= \sinh^2 x + \sinh^2 y \, (\cosh^2 x - \sinh^2 x) \\
&= \sinh^2 x + \sin^2 y \\
&= 1/2 \left[(\cosh 2x - 1) + 1 - \cos 2y\right] \\
&= 1/2(\cosh^2 x - \cos 2y).
\end{aligned}
$$

Now

$$
\begin{aligned}
A &= \frac{1}{2} \int_0^{\frac{\pi}{4}} \int_0^{2} \left[\cosh^2 x - \cos 2y\right] dx\, dy \\
&= \frac{1}{2} \int_0^{\pi/4} \left[\frac{\sinh 4}{2} - 2 \cos 2y\right] dy \\
&= \frac{1}{4} \left[y \sinh 4 - 2 \sin 2y\right] \Big|_0^{\pi/4} \\
&= \frac{1}{4} \left[\frac{\pi}{4} \sinh 4 - 2\right] \\
&= \frac{\pi \sinh 4 - 8}{16}.
\end{aligned}
$$

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