SPECTRAL ANALYSIS IN HILBERT SPACES

A THESIS

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CHAPTER I

ELEMENTARY THEORY OF COMPACT OPERATORS

1.1 Basic Topological Concepts:

Before presenting special topics concerning spectral theory in Hilbert Spaces, we shall introduce several preliminary definitions and lemmas which shall be referred to throughout the thesis.

Definition 1.1.1: The finite set M is called an ϵ-net for the set E if there exists for every point x in E a point F in M such that $(x, F) < \epsilon$. If ϵ-net exists for E, then E is called totally bounded.

Lemma 1.1.2: A normed linear space X is a metric space with the metric defined by

$$\rho(x,y) = \|x-y\|$$

Lemma 1.1.3: A sequentially compact subset of a metric space is totally bounded.

Lemma 1.1.4: If Y is a compact set of a metric space X, then Y is separable.

Theorem 1.1.5: A necessary and sufficient condition for a subset E of a Metric space X to be compact is that for each $\varepsilon > 0$, there exists in X a finite ε-net for E. The condition is also sufficient, if X is a complete space.

Theorem 1.1.6: For any two elements in a Hilbert Space H,

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Lemma 1.1.7: Let x be in a Hilbert Space X and let P be a projection operator. If $x \perp H_0$, then $Px=0$.

1.2 Completely Continuous Operators

Definition 1.2.1: A continuous operator U mapping a normed linear space X into a normed linear space Y is called completely continuous if it transforms every bounded set in X into a compact set in Y.

Theorem 1.2.2: If U is a completely continuous operator mapping the normed linear space X into the normed linear space Y, then the range of U is separable.

Proof: Let S be the set $U(K_n)$ in the space Y, where K represents the sphere in X with center at 0 and with radius r. Hence

$$S_n = \{y = U(x): \ x \in K_n\},$$

Since

$$X = \bigcup_{n=1}^{\infty} K_n ,$$

$$= U(X) = U\left(\bigcup_{n=1}^{\infty} K_n\right) = \bigcup_{n=1}^{\infty} \left(U(K_n) \right) = \bigcup_{n=1}^{\infty} S_n .$$

But S_n is compact by compactness of U. Therefore S_n has an ϵ-net F_n. Let $D = \cup F_n$. Then D is countable and dense. Hence S_n is separable. Therefore $X = \bigcup_{n=1}^{\infty} S_n$ is separable.

Theorem 1.2.3: Let $U = \alpha U_1 + BU_2$ be a linear combination of completely continuous operators. Then U is completely continuous.

Proof: Let E be a bounded set. Let $\{y_n\} \subset U(E)$. Then,

$$y_n = \alpha(U_1(x_n) + BU_2(x_n), \text{ where } x_n \in E$$
$$\text{for } n = 1, 2, \ldots.$$

Since U_1 and U_2 are completely continuous, we can choose from the sequences $\{U_1(x_n)\}$, $\{U_2(x_n)\}$ convergent subsequences $\{U_1(x_{n_j})$, $U_2(x_{n_{j_i}})$ respectively. Hence the sequence $\{U_1(x_{n_{j_i}})\}$ is convergent. This implies that U (E) is compact. Hence U is completely continuous.

Let U and V be linear operators from X into Y and Y into Z, respectively. Let one of these operators be completely continuous then VU is also completely continuous.

Proof: Let U be completely continuous and V continuous. Let E be bounded. Let x_n be in E, $n = 1, 2, 3 \ldots$.

Since U is completely continuous, we choose from the sequence $\{U(x_n)\}$ a convergent subsequence $U(x_{n_k})$.

Let

$$U(x_{n_k}) \longrightarrow y_0, \quad \text{for } y_0 \in Y.$$

Then

$$VUx_{n_K} = V\ (Ux_{n_K}).$$

Since V is continuous

$$V\ (Ux_{n_K}) \rightarrow V\ (y_0).$$

Hence V (U(E) is sequentially compact. Thus V U is completely continuous.

Let V be completely continuous. Let E be a bounded set. Since, U is a bounded linear operator, E is transformed into a bounded set by U, i.e., U (E) is bounded. Since V is completely continuous, VU(E) is compact. Hence VU is completely continuous.

Theorem 1.2.4: Let $\{U_n\}$ be a sequence of bounded linear operators from a complete space X into a space Y such that $U_n \rightarrow U$ (in the space of operators $[X \rightarrow Y]$). If the U_n,(n=1,2,...) are completely continuous, then U is also completely continuous.

Proof: Let S represent the unit sphere of the space X. It is only necessary to show that U (S) is compact.

Since U_n is convergent, for $\epsilon > 0$ there exists $n_0 > 0$ such that

(1.2.4) $\quad \| U_{n_0} - U \| \leq \epsilon/2.$

Let $y = Ux$ where $x \varepsilon S$. Let $y_{n_0} = U_{n_0} x$. Then using (1.2.4) we have

$$(1.2.5) \quad \|y-y_{n_0}\| = \|U(x)-U_{n_0}(x)\| \leq \|U-U_{n_0}\| \; \|x\| < \epsilon/2$$

Since U_{n_0} is compact, $U_{n_0}(S)$ is compact. But then there exists an $\epsilon/2$-net $F_\epsilon = \{z_1, z_2, \ldots z_n\}$ of $U_{n_0}(S)$. Hence there exists $z_{i_0} \in F_\epsilon$ such that

$$(1.2.6) \qquad \|y_{n_0} - z_{i_0}\| < \epsilon/2 .$$

But

$$\|y-z_{i_0}\| \leq \|y-y_{n_0}\| + \|y_{n_0}-z_{i_0}\| < \epsilon/2 + \epsilon/2 = \epsilon$$

by (1.2.5) and (1.2.6).

Thus F_ϵ is an ϵ-net for $U(S)$. Therefore $U(S)$ is compact by theorem (1.2.3) Hence U is completely continuous.

CHAPTER II

BASIC CONCEPTS OF SELF-ADJOINT OPERATORS

2.1 Adjoint Operators:

Definition 2.1.1: Let X, Y be Hilbert Spaces. Let U be a bounded linear operator from X into Y and let y be in Y. Define

$$x'(x) = (Ux,y).$$

x' is clearly a linear functional.

Moreover,

$$|x'(x)| = |(Ux\ y)| \leq \|Ux\|\ \|y\| \leq \|U\|\ \|y\|\ \|x\|.$$

Hence

$$\|x'\| \leq \|U\|\ \|y\|$$

Thus x' is bounded and $x' \in X$. By Riesz-Frechet Theorem there exists a unique z in X such that $\|x'\| = \|z\|$ and $x'(x) = (x,z)$ for all x in X. Let U^* be a mapping from Y into X defined by

$$U^* y = z$$

Thus

(2.1.2) $(Ux,y) = (x,U^*y)$ for all $x \in X$, $y \in Y$.

U^* is called the adjoint of U.

<u>Theorem 2.1.3</u>: The adjoint operator U^* mapping. Y into X is a linear operator and

$$\| U^* \| = \| U \|$$

<u>Proof</u>: (a) By (2.1.2)

$$(Ux, y_1 + y_2) = (x, U^*(y_1 + y_2))$$

But also

$$\begin{aligned}(Ux, y_1 + y_2) &= (Ux, y_1) + (Ux, y_2)\\ &= (x, U^* y_1) + (x, U^* Y_2)\\ &= (x, U^* y_1 + U^* y_2).\end{aligned}$$

Hence

$$(x, U^*(y_1 + y_2)) = (x, U^* y_1 + U^* y_2).$$

Therefore

$$(x, U^*(y_1 + y_2)) - (x, U^* y_1 + U \quad U y_2) = 0$$

This implies

$$(x, \boxed{U^*(y_1 + y_2)} - (U^* y_1 + U^* y_2)) = 0 \quad \text{for all} \quad x \in X.$$

Therefore,

$$U^*(y_1 + y_2) - (U^* y_1 + U_2^* y_2) = 0$$

Consequently,

$$U^*(y_1+y_2) = U^*y_1 + U^*y_2$$

(b) We know

$$(Ux, \alpha y) = (x, U^*(\alpha y)).$$

Also

$$(Ux, \alpha y) = \bar{\alpha}(Ux, y) = \bar{\alpha}(x, U^*y)$$
$$= (x, \alpha U^* y).$$

Hence

$$(x, U^*(\alpha y)) = (x, \alpha U^*y).$$

This implies

$$U^*(\alpha y) = \alpha U^*y.$$

Therefore U^* is a linear operator.

We now show that $\|U^*\| = \|U\|$.

Putting $X = U^* y$ in (2.1.2), we get

$$(U^*y, U^*y) = (UU^*y, y)$$

Hence

$$\|U^*y\|^2 = |(UU^*y, y)|$$
$$\leq \|U U^*y\| \; \|y\|$$
$$\leq \|U\| \; \|U^*y\| \; \|y\| .$$

Therefore

$$\|U^*y\| \leq \|U\| \; \|y\| .$$

Thus

$$\|U^*\| = \|U\| .$$

(c) Putting $y = Ux$ in (2.1.2) we get

$$(Ux, Ux) \quad (x, U^*Ux)$$

or

$$\|Ux\|^2 = |(x\ U^*Ux)| \leq \|x\|\ \|\ U^*Ux\|$$
$$\leq \|x\|\ \|U^*\|\ \ \|Ux\|.$$

Hence

$$\|Ux\| \leq \|x\|\ \|U\|$$

This implies

$$\|U\| \leq \|U^*\|.$$

Thus

$$\|U^*\| = \|\ U\|.$$

<u>Definition 2.1.4</u>: Denote the <u>second adjoint</u> of U by U^{**} and define it in the same manner as the adjoint U^* with $U^{**}: X \longrightarrow Y$.

<u>Theorem 2.1.5</u>: $U^{**} = U$.

<u>Proof</u>: $(U^*y, x)\ \ (y, U^{**}x)$.

Also

$$(y, Ux) = \overline{(Ux, y)} = \overline{(x, U^*y)} = (U^*y, x).$$

Therefore

$(y, Ux)\ \ (y, U^{**}x)$ for all $x \in X$, $y \in Y$.

Hence

$$U = U^{**}.$$

2.2 Eigenvalues

<u>Definition 2.2.1</u>: An <u>eigenvalue</u> of an operator U is a number λ such that there exist an element $x_0 \neq 0$ with the property

(2.2.2) $Ux_0 = \lambda\ x_0$,

An element x for which (2.2.2) holds is termed an __eigenvector__ corresponding to the given eigenvalue λ. The eigenvectors corresponding to a given eigenvalue λ form a space called the __eigenspace__ H_λ.

__Lemma 2.2.3__: If U is self-adjoint then,

$$(Ux, y) = \frac{1}{4}\big[(U(x+y), x+y) - (U(x-y), x-y)\big] + \big[(U(x+iy), x+iy) - (U(x-iy), x-iy)\big]$$

__Proof__: Proof is trivial,

__Theorem 2.2.4__: If U is a self-adjoint operator, then

$$\|U\| = \sup_{\|x\|=1} |(Ux, x)| .$$

__Proof__: Let $Q = \sup |(Ux, x)|$ where $\|x\| = 1$. Then

$$|(Ux, x)| \leq \|Ux\| \; \|x\| = \|Ux\| \leq \|U\| \; \|x\| = \|U\| .$$

Therefore

(2.2.5) $Q = \sup |(Ux, x)| \leq \|U\|$.

We first observe that if U is self-adjoint then

(2.2.6) $(Ux, x) = (x, Ux) = \overline{(Ux, x)} \Longrightarrow (Ux, x)$ is real.

From lemma 2.2.3

$$(Ux, y) = \frac{1}{4}\big[(U(x+y), x+y) - (U(x-y), x-y)\big] + i\,\big[(U(x+yi), x+yi) - (U(x-yi), x-yi)\big] .$$

Considering (2.2.6),

$$\begin{aligned} \text{Re } (Ux, y) &= \frac{1}{4}\big[(U(x+y), x+y) - (U(x-y), x-y)\big] \\ &\leq \frac{1}{4} Q \big[((x+y), x+y) - (x-y, x-y)\big] \\ &= \frac{1}{4} Q \big[\|x+y\|^2 + \|x-y\|^2\big] \end{aligned}$$

$$= \frac{1}{4} Q \left[2\|x\|^2 + 2\|y\|^2 \right]$$
$$= \frac{1}{2} Q \left[\|x\|^2 + \|y\|^2 \right] .$$

Let $\|x\| = 1$ and $y = Ux / \|Ux\|$,

Then

$$\|Ux\| = \mathrm{Re}\,(Ux, y) \leq \frac{1}{2} Q \left[\|x\|^2 + \|y\|^2 \right]$$
$$= \frac{1}{2} Q \cdot \left[1 + \frac{\|Ux\|^2}{\|Ux\|^2} \right]$$
$$= \frac{1}{2} Q \left[1 + 1 \right] Q$$

Therefore

(2.2.7) $$\|U\| \leq Q.$$

Hence

$$\|U\| = Q.$$

<u>Theorem 2.2.8</u>: The eigenvalues of the operator U are real.

<u>Proof</u>: Let λ be an eigenvalue. Then there exists $x_0 \neq 0$ such that

$Ux_0 = \lambda x_0$. Let $x = x_0 / \|x_0\|$. Then

$\|x\| = 1$ and $Ux = \lambda x$. But then

(2.2.9) $$(Ux, x) = (\lambda x, x) = (\lambda x, x) = \lambda (x, x)$$
$$= \lambda \|x\|^2 = \lambda .$$

Since U is a self-adjoint (Ux, x) is real and hence λ is real.

<u>Theorem 2.2.10</u>: Let H_{λ_1} and H_{λ_2} be eigensubspaces corresponding to different eigenvalues λ_1 and λ_2 of the operator U. Then H_{λ_1} is orthogonal to H_{λ_2}.

<u>Proof</u>: Let x be in H_{λ_1} and y in H_{λ_2}. Then

$$Ux = \lambda_1 x \quad \text{and} \quad Uy = \lambda_2 y.$$

Therefore if $\lambda_1 \neq 0$ then

$$\begin{aligned} \lambda_1(x,y) &= (\lambda_1 x, y) = (Ux, y) = (x, Uy) \\ &= (x, \lambda_2 y) = \bar{\lambda}_2(X,y) \\ &= \lambda_2(x,y). \end{aligned}$$

Thus

$$(\lambda_1 - \lambda_2)\ (x,y) = 0.$$

This implies $(x,y) = 0$, since $\lambda_1 \neq \lambda_2$ by asump-tion.

Therefore H_{λ_1} and H_{λ_2} are orthogonal.

<u>Theorem 2.2.11</u>: A completely continuous self-adjoint operator U has at least one eigenvalue.

<u>Proof</u>: If $U=0$ then $\lambda = 0$ is obviously the eigenvalue because

$$Ux_0 = \lambda x_0, \text{ for any } X_0 \neq 0.$$

Let $U \neq 0$ and define

$m = \inf\limits_{\|x\|=1} (Ux, x)$ and $M = \sup\limits_{\|x\|=1} (Ux, x)$.

Then by Theorem 2.2.4

$\|U\| = \operatorname{Sup} |(Ux, x)|$.

But if $|m| < M$, then $M \geq m \geq 0$ and

$|(Ux, x)| = (Ux, x) \in [m, M]$.

Therefore $\sup\limits_{\|x\|=1} |(Ux, x)| = M$.

Also, if $|m| > M$ implies $M < 0$.

Then $\sup\limits_{\|x\|=1} |(Ux, x)| = |m|$.

Hence

$\|U\| = \max [|m|, M]$.

Define

$$\lambda_1 = \begin{cases} m & \text{if } \|U\| = |m| \\ M & \text{if } \|U\| = M. \end{cases}$$

We show that λ is an eigenvalue of the operator U.

Let $\|U\| = M$. Then from the definition of M there exists a sequence $\{x_n\}$ with $\|x_n\| = 1$ such that

(2.2.12) $(Ux_n, x_n) \longrightarrow M = \lambda$.

We can extract from the sequence $\{Ux_n\}$ a convergent subsequence since U is completely continuous and $\{x_n\}$ is bounded.

Let $\{Ux_n\}$ denote this subsequence which converges to y_0. Then

$$\|Ux_n-\lambda_1 x_n\|^2 = \|Ux_n\|^2 - 2\lambda_1 (Ux_n,x_n) + \lambda_1^2$$
$$\leq \|U\|^2 - 2\lambda_1(Ux_n,x_n) + \lambda_1^2 \longrightarrow \|U\|^2 - 2\lambda^2+\lambda^2$$
$$= 0 ,$$

Hence

$$Ux_n-\lambda x_n \xrightarrow[n\to\infty]{} 0.$$

Therefore

$$x_n=\frac{1}{\lambda_1}[Ux_n-(Ux_n-\lambda x_n)]\to\frac{1}{\lambda}[y-0]$$
$$= \quad y_0/\lambda \qquad \text{since U is bounded.}$$

Let $x_0 = y_0/\lambda_1$. Hence $x_n\to x_0$

Since U is a continuous operator

$$Ux_n\to Ux_0.$$

Therefore

$$Ux_0= y_0=\lambda_1 x_0 \ .$$

Since $\|x_0\|=1$, $x_0\neq 0$.

Therefore, λ_1 is an eigenvalue.

<u>Definition 2.2.13</u>: Let M be a closed linear subspace of a Hilbert Space. Then every x in H can be written uniquely in the form $x = y + z$, where y in M, z in $M^{\perp}$. Point y is called the <u>"Projection"</u> of x in M, and the operator P given by $Px=y$ is called the "projection" on M. Let P_λ be the projection on the eigensubspace H_λ.

<u>Theorem 2.2.14</u>: Let U be a completely continuous self-adjoint operator, then the set of eigenvalues of U is not more than countable and

(2.2.15) $U = \sum_n \lambda_n P_{\lambda_n}$ where $\lambda_1, \lambda_2, \ldots$ are different eigenvalues of U and convergence is in operator norm.

Let λ be an eigenvalue of U.

Then

(2.2.16) $\lambda P_\lambda = U P_\lambda = P_\lambda U$,

since for $P_\lambda x$ in H_λ and any x in H,

$$UP_\lambda x = \lambda P_\lambda x$$

and $UP_\lambda = \lambda P_\lambda$ is self-adjoint, and hence P_λ and U are permutable.

Let

(2.2.17) $U_2 = U_1 - \lambda_1 P_{\lambda_1}$ where $U_1 = U$. Using (2.2.16) and letting $\tilde{P} = I - P_{\lambda_1}$, I being the identity operator then,

(2.2.18) $U_2 = \tilde{P}U_1 = U_1\tilde{P}_1$

hence U_2 is also self-adjoint. By Theorem 1.2.3 U_2 is also completely continuous and with (2.2.18) we have

$$\|U_2\| \le \|\tilde{P}_1 U_1\| \le \|\tilde{P}_1\| \; \|U_1\| \le \|U_1\| .$$

Theorem 2.2.11 applied to U_2 gives us its numerically greatest eigenvalue, call it λ_2.

Since $|\lambda_1| = \|U_1\|$ and $|\lambda_2| = \|U_2\|$,

$$|\lambda_1| \ge |\lambda_2|.$$

It remains to show that λ_1 is not an eigenvalue of the operator U_2.

Let λ_1 be an eigenvalue of U_2, then there is an element $x \neq 0$ such that

$$U_2 x = \lambda_1 x.$$

From (2.2.17)

(2.2.19) $U_1 x - \lambda_1 P_{\lambda_1} x = \lambda_1 x.$

Applying P_λ to both sides of the equation and using (2.2.12) we have

$$\lambda_1 P_{\lambda_1} x = P_{\lambda_1} Ux - \lambda_1 P_{\lambda_1}^2 x = UP_{\lambda_1} x - \lambda_1 P_{\lambda_1} x = 0.$$

Therefore substituting in equation (2.2.19)

$$U_1 x = \lambda_1 x.$$

Thus we have an element x in H_λ , where

$$\lambda = P_{\lambda_1} x = 0 .$$

But this contradicts the fact that $x \neq 0$. Hence λ_1 is not an eigenvalue of the operator U_2.

Now we show that every non-zero eigenvalue of the operator U_2 is an eigenvalue of U_1.

Let $\lambda \neq 0$ be an eigenvalue of U_2 and let X be a non-zero element such that $U_2 x = \lambda x$.

Then by (2.2.18)

(2.2.20) $\quad U_1 \tilde{P}_1 x = \lambda x.$

Applying $\tilde{P}_1$ we have

$$\tilde{P}_1 U_1 \tilde{P}_1 = \lambda \tilde{P}_1 x.$$

Also

$$\tilde{P}_1 U_1 \tilde{P}_1 = U \tilde{P}_1 x^2 = U_1 \tilde{P}_1 x = \lambda x. \quad \text{Therefore}$$

$$\lambda \tilde{P}_1 x = \lambda x$$

implies $\quad \tilde{P}_1 X = X.$

Using (2.2.19) this gives

$$U_1 x = \lambda x.$$

λ is therefore an eigenvalue of U .

Now let X be an eigenvector of U_1 corresponding to the eigenvalue λ and H_{λ}, H_λ be orthogonal for $\lambda \neq \lambda_2$

. By Lemma (1.1.7), $P_{\lambda_1} x = 0$.

Therefore

$$U_2 X = U_1 x - \lambda P_{\lambda_1} x = U_1 x = \lambda x.$$

Hence λ is an eigenvalue of U_2.

Let us assume that U_2 is not identically zero. Then we can construct an operator such that $\quad U_3 = U_2 - \lambda_2 P_{\lambda_2}$

We continue in this manner and get operators $U_1, U_2, \dots, U_n$ which are completely continuous and self-adjoint. These operators have eigenvalues $\lambda_1, \lambda_2, \dots \lambda_n$. They are defined such that

(2.2.21) $U_{k+1} = U_k - \lambda_k P_{\lambda_k} = U - \sum_{k} \lambda_j P'_{\lambda_j}$

for $K = 1, 2, \dots n-1$

and

$|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n|$

Further

(2.2.22) $\|U_k\| = |\lambda_k|$ for $K = 1, 2, \dots, n-1$.

We have already shown that these λ_k will be different eigenvalues of $U_1 = U$.

Let $U_n = 0$ for all n. Then by using (2.2.21) we have

$$U = \sum_{j=1}^{n-1} \lambda_j P'_{\lambda_j}.$$

If $U_n \neq 0$ for any $n = 1, 2, \dots$, we get a sequence of operators U_1, $U_2 \dots$ and their eigenvalues $\lambda_1, \lambda_2, \dots$ In this case we show that λ_n converges to zero. Suppose λ_n does not converge to zero, then

$$|\lambda_n| \geq \lambda_0 > 0 \quad \text{for all } n = 1, 2, \dots$$

Let x_n in H_{λ_n} be such that $\|x_n\| = 1$. The elements

x_n are orthogonal to each other. Using (2.2.22)

$$\|Ux_m - Ux_n\|^2 = \|\lambda_m x_m - \lambda_n x_n\|^2 = (\lambda_m x_m - \lambda_n x_n, \lambda_m x_m - \lambda_n x_n)$$
$$= |\lambda_m|^2 \|x_m\|^2 + 0 + 0 + |\lambda_n|^2 \|x_n\|^2$$
$$= 2\lambda_0^2 > 0 \qquad \text{for } m \quad n.$$

Hence the subsequence $\{Ux_n\}$ is not convergent and no subsequence is convergent. But this contradicts the fact that U is completely continuous. Since $\|U_n\| = |\lambda_n|$ for all n.

$$U_n \xrightarrow[n\to\infty]{} 0.$$

Hence using (2.2.21) we get

$$U = \sum_{k=1}^{\infty} \lambda P_{\lambda_k}.$$

Therefore (2.2.15) is established.

Now we show that U has no non-zero eigenvalues apart from $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$.

Let λ be a non-zero eigenvalue such that $\lambda \neq \lambda_1, \lambda_2, \ldots$.

Then using the already established (2.2.15) we have

$$\lambda x = \sum_k \lambda_k P_{\lambda_k} x.$$

The elements $P_{\lambda_k} x$ in H_{λ_k} are orthogonal to each other.

Therefore, the following holds:

$$\lambda P_{\lambda_m} x = \lambda_m P_{\lambda_m} x \qquad \text{for } m = 1, 2, \ldots$$

Since $\lambda = \lambda_m$ by lemma (1.1.7)

$$P_{\lambda_m} x = 0$$

which implies $x = 0$. This contradicts the assumption that $x \neq 0$. Hence there are no non-zero eigenvalues of U apart from $\lambda_1, \lambda_2, \ldots$ We have shown that the set of eigenvalues of a completely continuous operator U is not more than countable.

CHAPTER III

THE SPECTRUM OF SELF-ADJOINT OPERATORS

3.1 Operator Polynomials:

Definition 3.1.1: Let U be a self-adjoint operator in the Hilbert Space H and let the bounds of U be defined by

$m = \inf_{\|x\|=1} (Ux, x)$ and $M = \sup_{\|x\|=1} (Ux, x)$.

Let

((3.1.2) $\varphi(t) = c_0 + c_1 t + \ldots + c_n t^n$ for all scalars c

and define

(3.1.3) $\varphi(U) = c_0 I + c_1 U + \ldots + c_n U^n$.

$\varphi(U)$ is called the operator polynomial.

Lemma 3.1.4: Operator polynominals satisfy the following conditions:

i) If $\varphi(t)$ is a real polynomial, then $\varphi(U)$ is a self-adjoint operator.

ii) If $\varphi(t) = \alpha\varphi_1(t) + B\varphi_2(t)$, then $\varphi(U) = \alpha\varphi_1(U) + B\varphi_2(U)$.

iii) If $\varphi(t) = \varphi_1(t)\,\varphi_2(t)$, then

$\varphi(U) = \varphi_1(U)\, \varphi_2(U)$

iv) If $UV = VU$, then $\varphi(U)\, V = V\, \varphi(U)$.

Proof: i) Let $\varphi(t)$ be a real polynomial then consider

$\varphi(U) = C_0\, I + C_1\, U \;\ldots\; C_n\, U^n$.

Each operator $I, U, \ldots, U^n$ is self-adjoint.

$\varphi(U)$ is self-adjoint for $C_0, \ldots, C_n$ real numbers.

ii) Let $\varphi(t) = \alpha\varphi_1(t) + B\,\varphi_2(t)$.

Using (3.1.2) we have

$\varphi(t) = \alpha(C_0 + C_1 t + \ldots + C_n t^n) + B\,(S_0 + S_1 t + \ldots + S_n t^n)$

for scalars C and S.

Then by (3.1.3)

$$\begin{aligned}\varphi(U) &= \alpha(C_0 I + C_1 (U) + \ldots + C_n U^n) \\ &\quad + B\,(S_0 I + S_1 (U) + \ldots + S_n (U^n)) \\ &= \alpha(\varphi_1(U) + B\,\varphi_2(U).\end{aligned}$$

iii) Let $\varphi(t) = \varphi_1(t)\ \varphi_2(t)$, then

$\varphi(t) = (C_0 + C_1 t + \ldots + C_n t^n)\ (S_0 + S_1 t + \ldots + S_n t^n)$

for scalars C and S.

Then by (3.1.3)

$$\begin{aligned}\varphi(U) &= (C_0 I + C_1 (U) + \ldots + C_n U^n)\ (S_0 I + S_1 U + \ldots + S_n U^n) \\ &= \varphi_1(U)\, \varphi_2(U).\end{aligned}$$

iv) Let $UV = VU$. Then

$$\begin{aligned}\varphi(U)\, V &= (C_0 I + C_1 U + \ldots + C_n U^n)\, V \\ &= C_0 I\ (V) + C_1 UV + \ldots + C_n U^n V \\ &= C_0 V\ I + C_1 VU + \ldots + C_n VU^n \\ &= V\ (C_0 I + C_1 U + \ldots + C_n U^n) \\ &= V\, \varphi(U).\end{aligned}$$

<u>Lemma 3.1.5</u>: We have

(3.1.6) $\|\varphi(U)\| = \max\limits_{t\in[m,M]} |\varphi(t)|$.

<u>Proof</u>: Let $\theta(t) = |\varphi(t)|^2$.

Since $\varphi(U)$ is a self-adjoint operator

(3.1.7) $$\|\varphi(U)\|^2 = \sup_{\|x\|=1} (\varphi(U)x, \varphi(U)x) = \sup_{\|x\|=1} (\bar{\varphi}(U)\varphi(U)x, x) = \sup_{\|x\|=1} (\theta(U)x, x).$$

But $\theta(t) \leq \max\limits_{t\in[m,M]} |\varphi(t)|^2$

which implies

$\theta(U) \leq r^2 I$ for $r^2 = \max\limits_{t\in[m,M]} |\varphi(t)|^2$.

Therefore substituting in (3.1.7)

$\|\varphi(U)\|^2 = \sup\limits_{\|x\|=1} (\theta(U)x, x) \leq r^2$.

Section 3.2: <u>The Spectrum and Regular Values of a Self-Adjoint Operator</u>

<u>Definition 3.1.7</u>: A number λ is a point of the spectrum of self-adjoint operator U if there exists a sequence (X_n) such that

(3.2.2) $Ux_n - \lambda x_n \xrightarrow[n\to\infty]{} 0$, $\|x_n\| = 1$

for n = 1,2,...

We can use as another synonymous definition, λ is a point of the spectrum if

(3.2.3) $\inf\limits_{\|x\|=1} \|Ux - \lambda x\| = 0$.

The set of all such points is called the <u>spectrum</u> of U denoted by S_u.

By the definition of eigenvalue, every eigenvalue of U is an element in the spectrum, but the spectrum may contain points other than the eigenvalues of U.

Lemma 3.2.4: The bounds of U are points of its spectrum.

Proof: Let $0 \le m \le M$ and let $\lambda = M$. We have $\|U\| = \lambda$ and for $\|x\| = 1$

$$\|Ux - \lambda x\|^2 = (Ux - \lambda x, Ux - \lambda x) = \|Ux\|^2 - 2\lambda (Ux, x) + \lambda^2$$
$$= 2\lambda^2 - 2\lambda (Ux, x) \le 2\lambda [\lambda - (Ux, x)].$$

This gives us $\inf\limits_{\|x\|=1} \|Ux - \lambda x\|^2 \le 2\lambda [\lambda - \sup\limits_{\|x\|=1} (Ux, x)]$,

$$= 2\lambda [M - M] = 0.$$

By (3.2.3) λ is in the spectrum of U.

Lemma 3.2.5: The spectrum of an operator U is a closed set.

Proof: Let λ_1 be such that λ_1 is not in S_U. Then

$d = \inf \| Ux - \lambda_1 x\| > 0$.

Let $|\lambda - \lambda_1| < d/2$. Then

$$\inf_{\|x\|=1} \|Ux - \lambda x\| \ge \inf \| Ux - \lambda x\|$$
$$= \sup \|\lambda_0 x - \lambda x\| > d - \frac{d}{2} = \frac{d}{2} > 0.$$

Hence $\lambda \notin S_U$.

Lemma 3.2.6: Let $Q(t)$ be a real polynomial. Then the spectrum of the operator $Q(U)$ contains all points μ of the form $\mu = Q(\lambda)$ for λ in S_U.

Proof: Let μ be a real number and consider the equation

$$Q(t) = \mu$$

with $t_1, t_2, \ldots, t_s$ as all the roots of this equation.

Hence, $\varphi(U) - \mu I$ can be expressed in the following manner:

(3.2.7) $\varphi(U) - \mu I = C \, (U - t_1 I) \, (U - t_2 I) \ldots (U - t_s I)$.

Let λ be in S_u. Then there is a sequence $\{x_n\}$ of elements such that

$\|x\| = 1$ and

$U x_n - \lambda x_n \longrightarrow 0$.

Put $t_s = \lambda$, and $\mu = \varphi(\lambda)$ in (3.2.7). Then,

$\varphi(U)\, x_n - \mu x_n = C \, (U - t_1 I) \, (U - t_2 I) \ldots (U x_n - \lambda x_n) \underset{n \to \infty}{\longrightarrow} 0$.

Therefore μ is a point in the spectrum of $\varphi(U)$.

Now, we assume that none of the t_k belong to S_u, then

$\delta_s = \inf_{\|x\|=1} \|U x - t_s x\| > 0$.

Let $\|x\| = 1$ and $\|y\| \geq \delta_s$. Then

$\|U x - t_s x\| = \delta_s$. Then $(U - t_{s-1} I)\,(U x - t_s x) : \|x\| = 1$

$\subset \{ S U_y - t_{s-1} y : \|y\| \geq \delta_s \}$.

Hence $\inf \, (U - t_s I)\,(U x - t_s x) \geq \inf_{\|y\| > \delta_s} (U - t_s I) = \delta_{s-1} > 0$.

We continue in this manner and we see that:

$\delta_1 = \inf_{\|x\|=1} \|\varphi(U)\, x - U x\| = 0$.

Therefore $\mu = \varphi(t_k)$ for $k = 1, 2, \ldots, S$ are not in the spectrum of $\varphi(U)$.

<u>Lemma 3.2.8</u>: Let $\varphi(t)$ be a polynomial then

$\|\varphi(U)\| = \max_{t \in S_u} |\varphi(t)|$.

<u>Proof</u>: Since U is self-adjoint, we have

(3.2.9) $\|\varphi(U)\|^2 = \sup_{\|x\|=1} (\varphi(U)\, x, \varphi(U)\, x)$

$= \sup_{\|x\|=1} (\bar{\varphi}(U)\, \varphi(U)\, x, x)$

$= \sup_{\|x\|=1} (\Psi(U)\, x, x$ where

$$\psi(t) = |\varphi(t)|^2.$$

(U) .

Hence $\|\varphi(U)^2\|$. is an upper bound of the operator $\Psi(U)$.

The upper bound of a positive operator $\Psi(U)$ is the same as the least upper bound of $S\,\Psi(U)$

(3.2.10) $\quad M\Psi(U) = \sup S\,\Psi(U)$.

Applying Lemma 3.2.6 we have

(3.2.11) $\quad \sup S\,\Psi(U) = \sup\limits_{t\in S_u} \Psi(SU) = [\sup\limits_{t\in S_u} |\varphi(t)|]^2$.

sup ( ) .

Using equations (3.2.9), (3.2.10), and (3.2.11) we get

$$\|\varphi(U)\|^2 = \sup\,(\Psi(U)x, x) = \sup S\,\Psi(U)$$
$$= [\sup_{t\in S_u} |\varphi(t)|]^2.$$

Therefore since S_μ is closed, sup is attained. Hence,

$$\|\varphi(U)\| = \sup_{t\in S_u} |\varphi(t)| = \max_{t\in S_u} |\varphi(t)|.$$

<u>Theorem 3.2.12</u>: Let $\varphi(t)$ be a continuous function in $[m, M]$.

Then,

$$\|\varphi(U)\| = \max_{t\in S_u} |\varphi(t)|.$$

<u>Proof</u>: Consider a sequence $\{\varphi_k(t)\}$ of polynomials. Let $\{\varphi_k(t)\}$ be uniformly convergent to $\varphi(t)$. Then using (3.2.8) we get

$$\varphi_k(U) = \max_{t\in S_u} |\varphi(t)|.$$

Taking the limit of both sides as $n \longrightarrow \infty$ we have

$$\|\varphi(U)\| = \max_{t\in S_u} |\varphi(t)|.$$

<u>Definition 3.2.13</u>: A complex number λ is a <u>regular value</u> of U if it does not belong to the spectrum of U.

<u>Theorem 3.2.14</u>: If λ is a regular value of the operator U, then there exist in the Hilbert Space H the inverse bounded linear operator R defined by

(3.2.15) $R_\lambda = [U - \lambda I]^{-1}$.

Also if such an operator R as defined in equation (3.2.15) exists then λ is a regular value.

<u>Proof</u>: We show that if λ in a regular value then R_λ exists. Let λ be a regular value and define a function S_λ on S_μ as follows

(3.2.16) $S_\lambda \ (t) \ \dfrac{1}{t-\lambda}$.

Let $R_\lambda = S_\lambda \ (U)$.

From (3.2.16) we have

$(t-\lambda) \ S_\lambda \ (t) = 1$ for $t \in S_\mu$.

$(U - \lambda I) \ S_\lambda (U) = (U - \lambda I) \ R_\lambda = R_\lambda (U - \lambda I) = I$.

Therefore,

$$R = [U - \lambda I]^{-1}.$$

We now show that if the inverse bounded linear operator R exists, then λ is a regular value.

Let inverse operator $R_\lambda = [U - \lambda I]$ exists.

Let $\|X\| = 1$. Then

$R_\lambda \ (U - \lambda I) \ x\| = \|x\| = 1$.

Therefore, since R_λ is a bounded linear operator,

$1 = \| R_\lambda \ (U - \lambda I)x\| \leq \|R_\lambda\| \ \|U_\lambda - \lambda x\|$.

Hence,

$$\inf_{\|x\|=1} \|Ux-\lambda x\| \geq \frac{1}{\|R_\lambda\|} > 0. \qquad 0.$$

Hence $\lambda \notin S_U$. Therefore λ is a regular value of U.

<u>Theorem 3.2.17</u>: Let $\varphi(t)$ be a continuous real function defined on S_U. Then the spectrum of the operator $\varphi(U)$ contains all points μ of the form

$\mu = \varphi(\lambda)$ for λ in S_U .

<u>Proof</u>: Let μ be a point outside the spectrum .

Let ψ be a continuous function defined by

$$\psi(t) = \frac{1}{\varphi(t)-\mu} \text{ for } t \text{ in } S_U.$$

We have $\Psi(U)$ defined by

$$\Psi(U) = [\varphi(U) - \mu I]^{-1}.$$

Using Theorem (3.2.6) we have μ is a regular value for $\varphi(U)$, i.e. $\mu \notin S_{\varphi(U)}$.

Let $\mu = \varphi(\lambda)$ for λ in S_U.

Consider a sequence $\{\varphi_n(t)\}$ of polynomials which is uniformly convergent on S_U to the function $\varphi(t)$. Then

$$\begin{aligned}\|\varphi(U)\,x - U\,(x)\| &= \|\varphi_n(U)\,x - \varphi_n(\lambda)\,x + \varphi(U)\,x - \varphi_n(U)\,x \\ &\quad + \varphi_n(\lambda)\,x - U\,(x)\|. \\ &\leq \|\varphi(U)\,x - \varphi_n(\lambda)\,x\| + \|\varphi(U)\,x - \varphi_n(U)\,(x)\| \\ &\quad + \|\varphi_n(\lambda)\,x - \mu\,(x)\| \\ &\leq \|\varphi(U)\,x - \varphi_n(\lambda)\,x\| + \|\varphi(U) - \varphi_n(U(x))\|\,\|x\| \\ &\quad + |\mu - \varphi_n(\lambda)|\,\|x\|.\end{aligned}$$

Applying Lemma (3.2.6) we get

$\inf \| \varphi_n(U)x - \varphi(\lambda)x \| = 0.$

Hence,

$\inf_{\|x\|=1} \| (U(x - \mu x \| \leq \| \varphi(U - \varphi_n(U) \| + \mu - \varphi_n(\lambda))$.

Taking the limit as $n \to \infty$ we have,

$\inf_{\|x\|=1} \| (U)x - \mu x \| = 0.$

Therefore μ belongs to the spectrum of U.

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