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MOTION OF THE FREE END OF A SPIRAL SPRING

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CHAPTER I
INTRODUCTION

Problem. — We shall discuss in this paper the motion of a particle attached to one end of a spiral spring, the other end of which is fixed.

The spring will be looked upon as an ideal one, obeying Hooke's law for any amount of extension or contraction. Thus it may be replaced by an abstract force emanating from the fixed end and can be expressed mathematically by Hooke's formula for the relation between the tension and the elongation of disturbed elastic bodies. No other force is to act upon the particle at the end of the spring except the initial energy of projection and the tension in the spring.

Notation. — Let

\[ l = \text{the length of the undisturbed spring}, \]
\[ r = \text{the length of the disturbed spring}, \]
\[ n = \text{the modulus of elasticity}, \]
\[ T = \text{the tension in the spring}, \]
\[ m = \text{the mass of the particle}, \]
\[ \ddot{x} = \text{the acceleration of the particle}. \]
Hooke's law. - The increase in length of an elastic spring is proportional to its tension. According to Hooke's law, the following relation is readily seen to be true

\[ T = k(r-1), \]

where \((r-1)\) is the increase in length and \(k\) is the factor of proportionality.

The modulus of elasticity of a spring is defined as the force required to double its natural length. Hence we have

\[ n = k(2l-1) = kl, \]

or

\[ k = \frac{n}{l}. \]

Therefore

\[ T = \frac{n}{l}(r-1). \]  \(1\)

Equation (1) can be discussed under two different conditions:

(a) If \(r < 1\), (1) becomes

\[ T = \frac{n}{l}(1-r), \]

and \(T\), being negative, is an attractive force.

(b) If \(r > 1\), (1) becomes

\[ T = \frac{n}{l}(r-1), \]

and \(T\), being positive, is a repellant force.
CHAPTER II
PRELIMINARY EQUATIONS

Equations of Motion'. - The force, which was introduced in the last chapter, is always directed toward a fixed point, and for this reason it is a central force. Let us now set up the equations of motion for this force.

Let $T_x$, $T_y$, and $T_z$ be the components of $T$ parallel to the $x$, $y$, and $z$ axes respectively as shown in figure 1; also let $\lambda, \mu, \nu$ be its direction cosines. In figure 1 we see that

$$T_x = T \cos \alpha = \lambda T,$$
$$T_y = T \cos \beta = \mu T,$$
$$T_z = T \cos \gamma = \nu T.$$

But

$$\lambda = \frac{x}{r},$$
$$\mu = \frac{y}{r},$$
$$\nu = \frac{z}{r},$$

where $r$ is the radius vector and has the value $\sqrt{x^2 + y^2 + z^2}$.

We can now set up our equations of motion, equating the force to the product of the mass and the acceleration of the particle. Since $m$ is the mass of the particle, the equations of motion are

'MacMillan, Theoretical Mechanics, Page 265
\[ Mx'' = T \frac{x}{r}, \]
\[ My'' = T \frac{y}{r}, \]
\[ Mz'' = T \frac{z}{r}, \]

where \( x'', y'' \) and \( z'' \) are the components of the acceleration in the \( X, Y, \) and \( Z \) directions respectively. Throughout this discussion the primes denote differentiation as to \( t. \)

The differential equations (2) are of the sixth order; therefore six constants of integration are necessary for a complete solution. These six constants of integration can be regarded as being determined by the six initial coordinates of position and velocity, namely \( x_0, y_0, z_0, x'_0, y'_0, z'_0. \)

**Moment of Momentum Integrals.** Additional equations of motion of the particle are obtained by taking moments of the forces about the axes. Let \( O \) be an arbitrary point which is taken as the origin of a rectangular system of coordinates (Fig. 2).

Consider the force \( T \) at \( (x, y, z) \). This force may be replaced by its rectangular components \( T_x, T_y, \) and \( T_z. \) The introduction of two equal and opposite forces at \( B \) and also at \( O, \) each numerically

equal and parallel to $T_x$ at $A$, has no effect upon the system of forces. The forces $T_x$ at $A$ and $-T_x$ at $B$ form a couple whose axis is parallel to the $Y$ axis and whose moment is $T_x Z$. The forces $T_x$ at $B$ and $-T_x$ at $O$ form a couple whose axis is parallel to the $Z$ axis and whose moment is $-T_x y$.

The force $T_x$ at $A$ is thus replaced by

1. an equal and parallel force $T_x$ at $O$,
2. a couple $T_x z$ whose axis is parallel to the $Y$ axis, and,
3. a couple $T_y x$ whose axis is parallel to the $Z$ axis.

Similarly the force $T_z$ at $A$ is replaced by,

1. a force $T_z$ at $O$,
2. a couple $-T_z x$ whose axis is parallel to the $Y$ axis and,
3. a couple $T_y x$ whose axis is parallel to the $Z$ axis.

Similarly the force $T_z$ at $A$ is replaced by,

1. a force $T_z$ at $O$,
2. a couple $-T_z x$ whose axis is parallel to the $Y$ axis and,
3. a couple $T_y z$ whose axis is parallel to the $X$ axis.

Therefore the force $T$ at $A$ is replaced by

1. Its components $T_x$, $T_y$, $T_z$ at the point $O$,
2. a couple whose moment is $T_y z - T_y Z$ and whose axis is parallel to the $X$ axis,
3. a couple whose moment is $T_x z - T_z x$ and whose axis is parallel to the $Y$ axis.
(4) A couple whose moment is \( T_{xy} - T_{x}y \) and whose axis is parallel to the \( Z \) axis.

Since

\[
T_{x} = Mx',
\]
\[
T_{y} = My',
\]
and
\[
T_{z} = Mz',
\]

we can find other expressions for our three couples \( L, M, \) and \( N \) about the \( X, Y, \) and \( Z \) axes respectively. Hence

\[
L = T_{zy} - T_{y}z = M(yz'' - zy'),
\]

and

\[
M = T_{xz} - T_{z}x = M(zx'' - xz'),
\]

and
\[
N = T_{yx} - T_{y}x = M(xy'' - yx').
\]

We must now evaluate these last three expressions for our particle. If we multiply the second equation of (2) by \(-z\) and the third by \(y\) and add, we obtain the equation

\[
M(yz'' - zy') = 0,
\]

and similarly

\[
M(zx'' - xz') = 0,
\]

and

\[
M(xy'' - yx') = 0.
\]

Integrating these equations, we obtain,

\[
M(yz'' - zy') = mc',
\]
\[
M(zx'' - xz') = mc_{2},
\]
\[
M(xy'' - yx') = mc_{3},
\]

the right members being the constants of integration.

The components of the momentum of the particle are

\[
mx', my', mz',
\]
momentum being defined as the product of mass and velocity. Hence the equations in (3) are the components of the moment of momentum, which is a vector since momentum is a vector quantity. Each of the three components is constant, for this reason the momentum vector itself is a constant. On the basis of this fact and because the force acting on the particle is a central force, the three equations in (3) can be summed up in the single statement: "The moment of momentum of the particle is constant."

**Energy Integral.** - On multiplying the first equation of (2) by \(x'\), the second by \(y'\), the third by \(z'\) and then adding the three equations, we get

\[
m(x'x'' + y'y'' + z'z'') = T \frac{xx' + yy' + zz'}{r} \tag{4}
\]

since

\[
x^2 + y^2 + z^2 = r^2,
\]

and on differentiation we have

\[
x x' + y y' + z z' = r r'.
\]

The left member of (4) is an exact derivative, that is

\[
x' x'' + y' y'' + z' z'' = \frac{1}{2}(x'^2 + y'^2 + z'^2).
\]

The right member of (4) is also an exact derivative if

\(T\) is a function of \(r\) alone, that is if \(T\) does not depend upon \(r', e, e',\) or \(t\). In this event there exists a potential function \(U (r)\), such that

---

*MacMillian, Theoretical Mechanics, Page 266*
\( f(r) = \frac{\partial U}{\partial r} \)
and equations (4) can be integrated. The result of integration is

\[
\frac{1}{2}m(x'^2 + y'^2 + z'^2) - U(r) = c,
\]

(5)
equation (5) is our energy integral or \textit{via viva} integral. The first term of (5) namely

\[
\frac{m}{2}(x'^2 + y'^2 + z'^2) = \frac{mv^2}{2},
\]
is the kinetic energy of the particle; the second term

\[-U(r)\]
is the potential energy. Thus, from equation (5), "the kinetic energy of the particle plus its potential energy is a constant, which is denoted by the letter \( c \)."

\textbf{Motion in a Plane.} - Let us multiply the first equation of (5) by \( x \), the second by \( y \), the third by \( z \) and then add. We obtain

\[
c_1 x + c_2 y + c_3 z = 0.
\]

(6)
This is the equation of a plane through the origin and through the point \((x, y, z)\). The coordinates of the particle are \( x, y, \) and \( z \) and hence the particle always lies in a fixed plane which passes through the origin.

It is evident that the motion of the particle lies in a fixed plane since a plane is determined by the position of the center of force, the initial position of the particle, and the initial direction of the motion of the particle. The force acting upon the particle also lies in this plane, so that there

\textit{MacMillian, Theoretical Mechanics, page 267}
is nothing to make the particle depart from it.

For the sake of simplicity we will choose the plane of motion as the XY plane. Then
\[ Z = 0, \quad c_1 = c_2 = c. \]
The values for \( c_1 \) and \( c_2 \) can be verified by substituting \( z = 0 \) in equation (3). The three equations of (2) now reduce to
\[ \begin{align*}
mx'' &= T \frac{x}{r} \\
my'' &= T \frac{y}{r},
\end{align*} \]
(7)
two equations of the fourth order. The moment of momentum integral (3), after the removal of the factor \( m \) and the replacement of the letter \( c_3 \) by the letter \( h \), becomes
\[ xy' - yx' = h, \]
and the energy integral becomes
\[ \frac{1}{2} m(x'^2 + y'^2) - U(r) = 0. \]
CHAPTER III
THE DIFFERENTIAL EQUATIONS OF THE ORBITS

The Equations of Motion. — One form of our equation of motion is acceleration = force ÷ mass. From equation (1)

\[ T = \frac{m}{r} (r-1) \]

is our expression for the force. Since the mass of our particle is \( m \) and the acceleration is \( F \), we have

\[ F = \frac{m}{lm} (r-1). \]  

(8)

Now let

\[ \frac{n}{lm} = -\gamma^2. \]

Equation (8) then becomes

\[ F = -\gamma^2(r-1). \]

Take the value of \( \gamma \) as unity and (8) becomes

\[ F = -(r-1). \]  

(9)

The Equations of Motion in Polar Coordinates. — The radial and transverse components of acceleration of a point moving in a curve may be obtained by projecting the rectangular components of acceleration of the point on the radius vector and on a perpendicular to the radius vector. The \( x \) and \( y \) components of the velocity may be obtained by differentiating the relations \( x = r \cos \theta \), and \( y = r \sin \theta \) (Fig. 3). Thus
\[ \frac{dx}{dt} = \frac{dr}{dt} \cos \phi - r \sin \phi \frac{de}{dt}, \]
\[ \frac{dy}{dt} = \frac{dr}{dt} \sin \phi + r \cos \phi \frac{de}{dt}. \]

Differentiating these again, we have
\[ \frac{d^2x}{dt^2} = \frac{d^2r}{dt^2} \cos \phi - 2 \frac{dr}{dt} \frac{de}{dt} \sin \phi + \left( \frac{de}{dt} \right)^2 \frac{d^2\phi}{dt^2} + r \cos \phi \frac{de}{dt} \frac{d^2\phi}{dt^2}, \]
and
\[ \frac{d^2y}{dt^2} = \frac{d^2r}{dt^2} \sin \phi + 2 \frac{dr}{dt} \frac{de}{dt} \cos \phi - r \sin \phi \frac{d^2\phi}{dt^2} + r \cos \phi \frac{de}{dt} \frac{d^2\phi}{dt^2}. \]

Since the rectangular components of the acceleration are \( \frac{d^2x}{dt^2} \) and \( \frac{d^2y}{dt^2} \), the sum of the projections of the rectangular components of acceleration of the point on the radius vector is
\[ \frac{d^2x}{dt^2} \cos \phi + \frac{d^2y}{dt^2} \sin \phi, \]
and on a perpendicular to the radius vector is
\[ -\frac{d^2x}{dt^2} \sin \phi + \frac{d^2y}{dt^2} \cos \phi. \]

Substituting the values of \( \frac{d^2y}{dt^2} \) above in the last obtained expressions, we get for the radial acceleration
\[ Fr = \frac{d^2r}{dt^2} - r \left( \frac{de}{dt} \right)^2 - r'' - r e'^2, \tag{10} \]
and for the transverse acceleration
\[ Fe = 2 \frac{dr}{dt} \frac{de}{dt} + r \frac{d^2e}{dt^2} = 2 r' e' + r e'' = \frac{1}{r} (r^2 e')', \tag{11} \]

All of the motion of the particle is along the radius vector, and for this reason the components of acceleration along the perpendicular to the radius vector is zero. Hence
\[ \frac{1}{r} \frac{1}{r} (r^3 e')' = 0. \]

Since \( \frac{1}{r} \) cannot be zero, then
\[ (r^2 e')' = 0, \]
or
\[ r^2 e' = h. \tag{12} \]
From equations (9) and (10) we have
\[ r^n - r e^{\epsilon^2} = -(r-1), \text{ or } r^n - r e^{\epsilon^2} + (r-1) = 0. \] (13)

**Areal Velocity**: We will now discuss the character of the constant \( h \) of equation (12). Let \( A \) denote the area which has been swept over by the radius vector, starting from some convenient initial position, say \( \epsilon = 0 \), (Fig. 4). In the figure let \( P_1 \) and \( P_2 \) be two positions of the particle which subtend an angle \( \Delta \epsilon \) at the origin. Then aside from terms which depend upon the curvature of arc \( P_1 P_2 \), \( \Delta A \) is equal to the area of triangle \( 0 P_1 P_2 \), \( \Delta A \) is equal to the area of triangle \( 0P_1P_2 \).

Hence
\[
\Delta A = \frac{1}{2} (r + \Delta r)(r \sin \Delta \epsilon)
\]
or
\[
\frac{\Delta A}{\Delta t} = \frac{1}{2} (r + \Delta r) \cdot r \sin \frac{\Delta \epsilon}{\Delta t} \cdot \frac{\Delta \epsilon}{\Delta t}.
\]

Since \( \lim_{\Delta \epsilon \to 0} \frac{\sin \Delta \epsilon}{\Delta \epsilon} = 1 \), and since \( \Delta r \to 0 \) and \( \Delta \epsilon \to 0 \), as \( \Delta t \to 0 \), we have
\[
A' = \frac{1}{2} r^2 e'.
\]

---

'MacMillan, *Theoretical Mechanics*, page 269'
Since \( x = r \cos \theta \),
and \( y = r \sin \theta \),
then \( \tan \theta = \frac{y}{x} \).

Differentiating, we have
\[
e' \sec^2 \theta = \frac{xy' - yx'}{x^2}.
\]

But
\[
\sec^2 \theta = \frac{1}{\cos^2 \theta} = \frac{r^2}{x^2} = \frac{x^2 + y^2}{x^2};
\]
therefore
\[
e' = \frac{xy' - yx'}{r^2},
\]
and
\[
r^2 e' = xy' - yx'.
\]

From our moment of momentum integral, we know that
\[xy' - yx' = \hbar,\]
hence
\[2A' = r^2e' = xy' - yx' = \hbar.\]

The rate at which the radius vector sweeps over areas, \( A' \), is called the **Areal Velocity**. The constant \( \hbar \) is therefore twice the areal velocity.

We have seen that the differential equations of the orbit of the particle are:
\[
r^2e' = \hbar
\]
\[
r'' - r e'^2 + (r-1) = 0.
\]

We shall now eliminate \( \theta \) from these equations. From the first we have
\[
e' = \frac{\hbar}{r^2}.
\]
Substituting this value in the second equation, we have

\[ r^n = \frac{hs}{r^2} + (r-1) = 0, \]

or

\[ r^n = \frac{hs}{r^2} - (r-1); \]  \hspace{1cm} (14)

equation (14) can also be written as

\[ \frac{d^2r}{dt^2} = 2 \frac{hs}{r^3} - (r-1). \]

Multiplying both sides by \( 2dr/dt \), we have

\[ 2 \frac{dr}{dt} \frac{d^2r}{dt^2} = 2 \frac{hs}{r^3} \frac{dr}{dt} - 2(r-1) \frac{dr}{dt}; \]

integrating, we get

\[ \frac{(dr)^2}{dt} = -\frac{hs}{r^2} - r^2 + 2r + 1 + c, \]

or

\[ r'^2 = -\frac{hs}{r^2} - r^2 + 2r + 1 + c = -\frac{l_2}{r^2} (r^4 - 2s^2l - c r^2 + h^2) = v^2. \]  \hspace{1cm} (15)

The integral (15) is equivalent to our vis viva integral (5).

From equation (5) we see that if the radius vector is zero, the velocity is infinite, and if the radius vector is infinite the velocity is zero.
CHAPTER IV

SOLUTIONS OF THE DIFFERENTIAL EQUATION

A Particular Solution - \( h = 0 \)

In the last chapter we saw that our differential equations were reduced to (15) which is

\[ r'' = \frac{h^2}{r^3} -(r-1). \]

When \( h = 0 \), we get the following equation

\[ r'' = -(r-1). \quad (16) \]

We wish now to determine the type of motion represented by equation (16). The force itself is always directed toward the origin, and since it is negative when \( r \) is positive, and positive when \( r \) is negative, equation (16) is valid on both sides of the origin.

Simple Harmonic Motion. - The solution of equation (16) is simple, since we need only to ask the question: "what function of \( t \) reproduces itself, aside from a changed \( t \) and a constant term, when differentiated twice?" The most general answer is immediately found to be

\[ r + 1 = A \sin kt + B \cos kt. \]

Ours is a special case and our answer takes the form

\[ r + 1 = \sin t. \quad (17) \]

From the periodicity of the sine function, we may now infer that the motion of our particle as represented by equation (17) is simple harmonic motion. We may define simple harmonic motion
as the motion of the projection on a straight line of a point which moves with uniform speed in a circle, the projected motion being in the plane of the circle.

Since our motion is simple harmonic, it can be represented as in figure 5. We denote by $r$ the displacement of the free end of the spring from the fixed end $0$.

Equation (16), our equation of motion, takes on different forms with variations of $r$ and $l$, namely

(a) if $r > 1$, (16) becomes $r'' = -(r-1)$ and since $r''$ is negative, equation (16) represents a repellant force,

(b) if $r < 1$, (16) becomes $r'' = (1-r)$, and since $r''$ is positive, equation (16) represents an attractive force and

(c) if $r = 1$, the acceleration is zero.

It is evident that whenever we have a force at all, the force must fall under either condition (a) or (b). We can justly say therefore, that equation (16) holds for all points on $OA$ (Fig.5) only if $r$ changes sign when $l$ changes sign.

Our problem now divides into two problems: the physical problem and the mathematical problem. We will refer the motion to point $B$ of figure 5, and consider only such values of the displacement $Z$ with respect to the new origin as are defined by $|Z| < l$. This constitutes the physical problem. Then we shall make the problem purely a mathematical one by considering all
values of \( r \) defined by \( |r| < \infty \).

The Physical Problem. - In the physical problem we make the transformation,

\[
z = r - 1.
\]

Then

\[
z^\prime = r^\prime = -(r - 1) = -z.
\]  \( (18) \)

We are referring the motion to \( B \) as the pole, the fixed end of the string remaining at \( o \). Equation (18), being similar to equation (16), shows that the motion of the particle with respect to \( B \) is simple harmonic motion. From equation (17) it is evident that the period of the motion is the same as the period of the sine function. If we denote the period by \( P \), we have

\[
P = 2 \pi.
\]

Since there are no arbitrary constants involved in the value of \( P \) it is readily seen that the period is independent of the initial conditions.

Mathematical Problem. - Let \( r > 21 \). If the initial displacement of \( r \) is > 21, the particle will pass through \( B \). The motion is a modified type of simple harmonic motion, and is symmetrical with respect to \( B \). As has already been shown, all equations derived from \( z^\prime = -z \) will completely define the motion if \( 1 \) changes sign with \( r \).

We wish now to find the velocity and the period for our mathematical problem. From equation (16), we have

\[
r^\prime = -(r - 1) \text{ or } \frac{d^2r}{dt^2} = -(r - 1).
\]

Hence, multiplying by \( 2 \frac{dr}{dt} \) we have
\[ 2 \frac{dr}{dt} \frac{d^2r}{dt^2} = -c(r-1) \frac{dt}{dt}. \]

Integrating, we have

\[ \frac{(dr)^2}{(dt)} = -(r-1)^2 + c. \]

For definiteness let us start our particle from rest at the right of the origin where initial conditions are

\[ r_0 = a + 1, \]

since \( r = a + 1 \) when \( r = \frac{dt}{dt} = 0 \), then \( c = (a + 1 - 1)^2 = a^2. \)

Therefore

\[ \frac{(dr)^2}{(dt)} = a^2 - (r-1)^2, \]

or

\[ r' = \frac{dr}{dt} = \frac{\sqrt{a^2 - (r-1)^2}}{a}. \quad (19) \]

Our period may be obtained by integrating (19) and evaluating our constant from the initial conditions. Thus from (19)

\[ \frac{dr}{\sqrt{a^2 - (r-1)^2}} = dt, \]

and integrating, \( \sin^{-1} \left( \frac{r-1}{a} \right) + c = t. \)

Since \( r = a + 1 \) when \( t = 0 \), we have

\[ c = \sin^{-1} \left( \frac{a + 1 - 1}{a} \right) = \sin^{-1} \left( \frac{a}{a} \right) = \frac{\pi}{2}. \]

Therefore

\[ t = \sin^{-1} \left( \frac{r-1}{a} \right) + \frac{\pi}{2}. \]

This is the time for the particle to go from the end point of its path to the origin. The period, therefore, is \( 4t \), and we have

\[ P = 4t = 4 \sin^{-1} \left( \frac{r-1}{a} \right) + 2\pi. \quad (20) \]

It is obvious from equation (2) that the period of the mathematical problem depends on the initial conditions. The mathematical
period has a minimum value when \( r = 1 \) and the period in this case is the same as the period of the physical problem. Likewise the mathematical period has a maximum value when \( r = a + 1 \) and the period in this case is double the period of the physical problem.

**Displacement.** - The displacement of the particle at any time can be found by integrating equation (19) and solving for \( r \). Thus we get

\[
    r = a \cos t + 1. \tag{21}
\]

We can verify, from equation (21), that \( r = a + 1 \), when \( t = 0 \).

This equation does not permit of a direct determination of the displacement at the end of any interval of time for the cosine is periodic. But by knowing, as we now do, the period of the motion we can locate the particle as being to the right or left of the origin and then determine \( r \) uniquely by means of equation (21.)

**Functional Variation.** - In the physical problem the velocity, acceleration, and displacement are continuous and simple periodic functions of the time. The same thing is true for the velocity and displacement in the mathematical problem; but the acceleration, although periodic is discontinuous at zero. It is this property of the acceleration that prevents a direct treatment of equation (16).

**The General Solution - \( h \neq 0 \)**

Our aim now is to evaluate equation (15), which is,

\[
    r^4 = -\frac{1}{r^2} (r^4 - 2r^3 - cr^2 + h^2),
\]

and thus obtain the general solution of equation (13).
Initial conditions. - Since the position of our polar axis is arbitrary, without loss of generality we may start our particle at an apse. It is to be understood now that \( h \) is not zero. Hence, as general initial conditions we take

$$ r(0) = r_0, $$

and

$$ r'(0) = 0. $$

Change of Linear Units. - Dividing equation (15) through by \( a \), (an arbitrary constant), we obtain

$$ \frac{r'^2}{a} = -\frac{1}{ar^2} \left( r^4 - 8r^3 - cr^2 + h^2 \right), $$

or

$$ \left( \frac{r'}{a} \right)^2 = -\frac{1}{ar^2} \left( \frac{r^4}{a^4} - \frac{8r^3}{a^3} - \frac{cr^2}{a^2} + \frac{h^2}{a^2} \right) $$

$$ = -\frac{r'^2}{a^2} + \frac{8}{a} \left( \frac{r}{a} \right)^3 + \frac{(-h^2)(a^2)}{r^2} + \frac{c}{a^2} \quad (22) $$

Now let \( \rho = \frac{r}{a} \), and (22) becomes

$$ (\rho')^2 = -\rho^2 + \frac{8}{a} \frac{1}{\rho} + \frac{(-h^2)(a^2)}{\rho^3} + \frac{c}{a^2} \rho $$

$$ = -c_1 \rho^2 - c_2 \rho + c_3 + c_4, \quad (23) $$

where

$$ c_1 = \frac{h^2}{a^4}, \quad c_2 = 1, \quad c_3 = \frac{21}{a}, \quad c_4 = \frac{c}{a^2}. $$

Since \( (\rho')^2 \) can never be negative (for we have a real problem and must deal with real quantities) we will set the right member of (23) equal to zero, and determine the range within which \( \rho \) is real and finite. Hence

$$ c_2 \rho^4 - c_3 \rho^3 - c_4 \rho^2 + c_1 = 0. \quad (24) $$

There are two variations in sign in equation (24), and therefore, according to Descartes' Rule of signs, there are two posi-
tive real roots or no positive real roots. Since we are dealing with real quantities we will say that we have two positive roots and since (25) is a biquadratic equation we also have two negative roots. We do not consider the question of all imaginary roots because we have a real problem, and for a specific case our coefficients can be so determined that all our roots are real.

The Periodicity of the Motion. — Let us now make the following notation:

\[ f(\rho) = c_2 \rho^4 - c_3 \rho^3 - c_4 \rho^2 + c_5. \]

Equation (23) then becomes

\[ \rho^2 (\rho')^2 = f(\rho). \] (25)

We may consider only those values of \( \rho \) which make \( \rho' \) positive, for \( \rho' \) would be imaginary if \( (\rho')^2 \) were negative. Since \( f(\rho) \) has four roots, we will call them \( \alpha_1, \alpha_2, \alpha_3, \) and \( \alpha_4. \) The graph of the function \( f(\rho) = 0 \) may be represented as in figure 6. We can, without the loss of generality, assume that \( f(\rho) \) is positive between \( \alpha_3 \) and \( \alpha_4. \) Having made this assumption, we can now find the time for \( \rho \) to vary from \( \alpha_3 \) to \( \alpha_4 \) from equation (25). Thus

\[ \rho' = \frac{\sqrt{f(\rho)}}{\rho}, \]

or

\[ \frac{d\rho}{dt} = \frac{\sqrt{f(\rho)}}{\rho}, \]

and

\[ \frac{\rho \, d\rho}{\sqrt{f(\rho)}} = dt. \]

Hence

\[ \tau = \int_{\alpha_3}^{\alpha_4} \frac{\rho \, d\rho}{\sqrt{f(\rho)}}. \] (26)
is the time for \( \psi \) to vary from \( \psi_3 \) to \( \psi_4 \). Moreover (26) is finite in accordance with the general theory of definite integrals.

Neither \( \psi_3 \) nor \( \psi_4 \) is a root of (25) as long as they are distinct; hence \( \psi \) is a periodic function of the time, and in general oscillates between two limiting values for which the left member of (25) vanishes. If these limiting values should happen to be equal, \( \psi \) is a constant; its value is the solution of (25) and the orbit is a circle.

**Apexes.** — The roots \( \psi_i \) are the apsidal distances of the orbit, i.e., they are the maximum and minimum distances of the particle from the origin. From this it follows that there are but two such distances; namely, the limiting values of \( \psi \). The number of apexes, however, may be infinite, for if the two limiting values of \( \psi \) should coincide, there would be an apex at every point of the orbit. In our problem \( \psi \) will always be defined by \( \psi_3 < \psi < \psi_4 \). The orbits, therefore, will be between the circumferences of two concentric circles whose center is the origin and whose radii, by equation (15) are finite.

**New constants of Integration.** — Equation (15) shows that it is expressible as an elliptic integral of \( \psi \). To reduce this elliptic integral to the standard form of Legendre it is necessary that we know explicitly the roots of our biquadratic equation. For this purpose we will introduce into equation (23) two new constants to replace \( a \) and \( h \). Such a step is valid provided the new constants are determined from the old and hence from the initial conditions.

Because of the scale factor \( a \) in equation 23, we may suppose that \( \psi \) vanishes for
\[ \rho = 1 + \varepsilon \]
and
\[ \rho = 1 - \varepsilon, \quad (27) \]
where \( \varepsilon \) is a real number or a pure imaginary. We shall not consider imaginary values of \( \varepsilon \), however, since for such values the elements of the motion become imaginary. Making the substitution (27) in (23) we get
\[ \omega = -c_2(1+\varepsilon)^4 + c_3(1+\varepsilon)^3 + c_4(1+\varepsilon)^2 - c_1, \]
and
\[ \omega = -c_2(1-\varepsilon)^4 + c_3(1-\varepsilon)^3 + c_4(1-\varepsilon)^2 - c_1. \]
We can now solve these two equations for \( c_1 \) and \( c_4 \) in terms of \( c_3 \), remembering that \( c_2 = 1 \). Our results are
\[ c_1 = (1-\varepsilon^2)^2 - c_3(1-\varepsilon^2)^2 = (1-\varepsilon^2)(1-\varepsilon^2)^2, \]
and
\[ c_4 = 2(\varepsilon^2 + 1) - c_2(\varepsilon^4 + 3). \]
If we now replace the \( c_1 \)'s by their values and put \( \frac{1}{\varepsilon} = \lambda \), we get
\[ h^2 = a^4(1-\lambda)(1-\varepsilon^2)^2 \]
\[ c = a^2 \left[ (2-3\lambda) + (2-\lambda)\varepsilon^2 \right]. \quad (28) \]

We shall now select \( a \) and \( \varepsilon \) as our arbitrary constants, since we have shown that by (23) they can be determined from the initial conditions.

**Range of New Constants.** — Since \( a \) is a scale factor connecting two radius vectors \( r \) and \( \rho \), it is sufficient to consider it positive. The second equation of (28) shows that \( a \) can never be infinite; and since \( h \) is not zero, it follows from the first equation of (28) that \( a > 1 \).

The square of the constant \( \varepsilon \) only, occurs in equation (28), so that only positive values of \( a \) need be considered. By the first equation of (28) \( a \) cannot equal unity, hence for the range
of our new constants, we may write

\[ 0 \leq \epsilon \leq 1 \]
\[ 1 \leq a < \infty . \]  

(29)

It follows from (29) that \( 0 < \lambda < 1. \)

**Roots of the Biquadratic.** — In equation (23) replace \( c_2 \) and \( c_3 \) by their original values and substitute for \( c_1 \) and \( c_4 \) the values obtained above. We then have

\[
(r')^2 - \frac{1}{\rho'} \left[ c_1 - 2\lambda \rho \right]^2 - \left\{ (2-3\lambda) + (2-\lambda) \epsilon^2 \right\} \rho^2 + (1-\lambda)(1-\epsilon^2)^2 .
\]

(30)

Now, since by hypothesis \( (r')^2 \) vanishes for \( r = 1 + \epsilon \) and \( r = 1 - \epsilon \), we know two of the roots of the expression within the brackets. Suppose the four roots of (30) are \( \alpha_1, \alpha_2, \alpha_3, \) and \( \alpha_4 \).

If we let \( \alpha_2 = 1 - \epsilon \), and \( \alpha_4 = 1 + \epsilon \) be two of the roots of the biquadratic, we can solve for the other two by methods of the theory of equations, and equation (30) becomes

\[
(r')^2 - \frac{1}{\rho'} \left[ c_1 - (1-\lambda) \right] \left[ c_1 - \sqrt{(1-\lambda)(1+\lambda)} \right] \left[ c_1 + \sqrt{(1-\lambda)(1+\lambda)} \right] \].

(31)

**Nature of Roots.** — Throughout our discussion the \( \alpha'_4 \) will be defined as follows:

\[ \alpha_1 = -(1-\lambda) - \sqrt{(1-\lambda)(1+\lambda)} , \quad \alpha_3 = 1 - \epsilon , \]
\[ \alpha_2 = -(1-\lambda) + \sqrt{(1-\lambda)(1+\lambda)} , \quad \alpha_4 = 1 + \epsilon . \]

(32)

Note that so far as affecting these roots is concerned, a change in the sign of \( \epsilon \) is equivalent to interchanging the subscripts between \( \alpha_3 \) and \( \alpha_4 \). Double roots occur if \( \epsilon = 0 \) and if \( \epsilon^2 = \lambda \).

Two of the roots always lie to the right of the origin and the other two to the left. For the order of magnitude we have

for \( \epsilon < \lambda, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) are conjugate imaginaries;

for \( \lambda < \epsilon < 1, \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 \).
CHAPTER V
GENERAL CHARACTER OF THE MOTION

Limiting Cases

1. e = 0. - Here \( \alpha_1 \) and \( \alpha_2 \) are conjugate imaginaries and \( \alpha_3 \)
and \( \alpha_4 \) are equal to unity (Fig. 7).
The orbit is a circle with radius equal to unity or \( r = a \). We will
call the initial velocity necessary to produce this orbit \( v_0 \).
Using this value of the initial velocity and equation (22) we
readily get

\[ \Theta = \frac{v_0}{a} t, \]

as the value of the polar angle. This value shows that the
angular velocity is uniform. Also, since the motion is in a
circle, we get for the period of the motion

\[ P = \frac{2\pi a}{v_0}. \]

2. e^2 = \lambda. - Here \( \alpha_1 = \alpha_2 \) and \( \alpha_3 \) and \( \alpha_4 \) are real and
distinct (Fig. 8).
By the initial conditions the motion is
oscillatory and the limiting values of
\( \rho \) are \( \alpha_3 \) and \( \alpha_4 \). Since two roots are
equal the problem is reducible, and
equation (21) becomes

\[ \int \frac{d\rho}{\sqrt{\rho^2 - (\alpha - \lambda)^2}} = \int \frac{\rho d\rho}{\sqrt{\rho^2 - (\alpha - \lambda)^2}}. \]
We will make the substitution
\[
\sin^2 \phi = \frac{(\alpha_4 - \alpha_3)(\alpha_4 - \alpha_2)}{(\alpha_4 - \alpha_3)(\alpha_4 - \alpha_2)}
\]  
(34)
in equation (33). Solving for \( \rho \), we have
\[
\rho = \alpha_2 + \frac{\alpha_3 - \alpha_2}{1 - \frac{\alpha_4 - \alpha_3}{\alpha_4 - \alpha_2} \sin^2 \phi}
\]
\[
\varrho = \frac{\alpha_4 - \alpha_3}{\alpha_4 - \alpha_2}, \quad \vartheta = \varrho_3 - \alpha_2.
\]

Making this substitution in equation (33), we have
\[
\int_{\varphi}^{\tau} dt = \int_{0}^{\phi} \frac{\sqrt{\alpha_4 - \alpha_3} \sqrt{(\alpha_4 - \alpha_3)(-\sin^2 \phi) + \varrho}}{\varrho} \left[ (\alpha_4 - \alpha_3)(-\sin^2 \phi) + \varrho \right]^{-1} d\phi
\]
\[
= M \int_{0}^{\phi} \left[ \alpha_2 + \frac{\varrho}{1 - \frac{\alpha_4 - \alpha_3}{\alpha_4 - \alpha_2} \sin^2 \phi} \right] d\phi,
\]
(35)
where
\[
M = \frac{2}{\sqrt{(\alpha_4 - \alpha_3)(\alpha_3 - \alpha_2)}}.
\]

The right side of equation (35) can be divided into two integrals, thus
\[
M \int_{0}^{\phi} \varrho_2 \ d\phi = M \varrho_2 \phi,
\]
and
\[
M \int_{0}^{\phi} \frac{\varrho d\phi}{1 - \frac{\alpha_4 - \alpha_3}{\alpha_4 - \alpha_2} \sin^2 \phi} = M (\alpha_3 - \alpha_2) \int_{0}^{\phi} \frac{\sin^2 \phi \ d\phi}{(\alpha_4 - \alpha_3) \left[ \frac{\alpha_4 - \alpha_3}{\alpha_4 - \alpha_2} + \tan^2 \phi \right]} = M (\alpha_3 - \alpha_2) \frac{1}{1 - \frac{\alpha_4 - \alpha_3}{\alpha_4 - \alpha_2} \tan \phi}
\]
\[
= 2 \tan^{-1} \left[ \frac{\alpha_3 - \alpha_2}{\alpha_4 - \alpha_2} \tan \phi \right],
\]
Equation (35) has a solution which is
\[
\tau = M \varrho_2 \phi + 2 \tan^{-1} \left[ \frac{\alpha_3 - \alpha_2}{\alpha_4 - \alpha_2} \tan \phi \right],
\]
(36)
The Period. — We can obtain the period of the motion of the
radius vector by putting $\phi = \pi$ in (36). The period, therefore, is

$$P = M \alpha_2 \pi - 2 \tan^{-1} o = (M \alpha_2, -2).$$

**The Polar Angle.** From the relation

$$\frac{d\phi}{dt} = \frac{d\phi}{de} \frac{de}{dt} = \frac{h}{a^2} \frac{d\rho}{de}$$

and equation (12), we get

$$\int_0^\phi d\theta = \frac{h}{a^2} \int_3^\infty \frac{d\rho}{\rho \sqrt{(\rho^2 - \lambda^2)}} \int_0^\infty \frac{d\rho}{\rho \sqrt{(\rho^2 - \theta^2)}}$$

(37)

Applying transformation (34) to this, we obtain

$$\int_0^\phi d\theta = \frac{M h}{\alpha_2 a^2} \int_0^\phi \left[ 1 - \frac{q}{\alpha_2 + \alpha_3 - \alpha_4 - \alpha_5} \right] d\phi$$

(33)

We can get a solution for equation (33) by breaking it up into two integrals and integrating as we did in the case of (36).

The result is

$$\theta = 2 \left[ \tan^{-1} \sqrt{\frac{\lambda_2 (\lambda_3 - \lambda_5)}{\lambda_3 (\lambda_4 - \lambda_5)}} \tan \phi - \frac{1}{\sqrt{4 - \lambda}} \phi \right]$$

(39)

The value of the polar angle at the end of one complete oscillation is obtained by putting $\phi = \pi$ in (39). Thus

$$\theta = 2 \left[ \pi - \frac{1}{\sqrt{4 - \lambda}} \right].$$

As $\lambda$ increases from zero towards unity, this value of $\theta$ decreases from $\pi$ towards 0.846 ....... 1. The end values are never obtained, however, since $0 < \lambda < 1$.

**General Cases**

1. $e^2 > \lambda$. - Here all the $\alpha_i$'s are real and distinct. The motion is oscillatory, there being no possibility of a circular solution (Fig. 9).
\[ 2e^2 < \lambda \]. Here \( \alpha_1 \) and \( \alpha_2 \) are conjugate imaginaries, \( \alpha_3 \) and \( \alpha_4 \) are real and distinct. The general character of the motion is the same as in 1.
CHAPTER VI

THE ELLIPTIC INTEGRALS OF THE SOLUTION

\( e^{-\lambda} \). All Roots real.

When all of the roots of (31) are real, the reduction of this integral to the normal form of Legendre may be accomplished by the substitution (34), which is

\[
\sin \phi = \frac{(a_{\gamma} - a_{\lambda})(a_{\gamma} - a_{\beta})}{(a_{\gamma} - a_{\beta})(a - a_{\gamma})},
\]

whence

\[
\rho = a_{\gamma} + \frac{q}{\sqrt{1 - \rho^2 \sin^2 \phi}},
\]

where

\[
q = a_{\gamma} - a_{\lambda}, \quad \beta = \frac{a_{\gamma} - a_{\beta}}{a_{\gamma} - a_{\lambda}}.
\]

From these relations and equation (31), we get

\[
\int_{0}^{\phi} \frac{\sqrt{\rho}}{\sqrt{1 - \rho^2 \sin^2 \phi}} d\phi = M \int_{0}^{\phi} \left[ \frac{a_{\gamma}}{\sqrt{1 - \rho^2 \sin^2 \phi}} + \frac{q}{(1 - \rho^2 \sin^2 \phi)} \right] d\phi,
\]

where

\[
M = \frac{2}{\sqrt{(a_{\gamma} - a_{\beta})(a_{\gamma} - a_{\lambda})}}, \quad \beta = \frac{a_{\gamma} - a_{\beta}}{a_{\gamma} - a_{\lambda}},
\]

and

\[
\kappa^2 = \frac{(a_{\gamma} - a_{\beta})(a_{\gamma} - a_{\lambda})}{(a_{\gamma} - a_{\beta})(a_{\gamma} - a_{\lambda})}.
\]

The two elliptic integrals in (40) are known respectively as the Legendre elliptic integral of the first and third kind. They give the value of \( t \) for any assigned value of \( \rho \). Hence the problem is completely solved for the case of real roots of (31). However, it will usually be found more desirable to have
\( \rho \) expressed as a function of \( t \), and such an expression can be found as a periodic solution of (40).

**The Period.** — Equation (40) can be expanded in a power series, thus

\[
\int_0^t \frac{dt}{M} = M \int_0^\phi \frac{1}{\sqrt{\left[\left(1 - K^2 \sin^2 \phi \right)^{n} + \left(1 - K^2 \sin^2 \phi \right)^{-n}\right]}} d\phi
\]

\[
= M \int_0^\phi \frac{1}{\sqrt{\left[\sum_{n=1}^{\infty} \frac{1}{4 \cdot 6 \cdot \cdots \cdot 2n - 1} K^{2n} \sin^{2n} \phi \right] + \left[1 + \sum_{n=1}^{\infty} \frac{1}{4 \cdot 6 \cdot \cdots \cdot 2n - 1} K^{2n} \sin^{2n} \phi \right]}} d\phi
\]

\[
= B \int_0^\phi \left[1 + \sum_{n=1}^{\infty} \frac{1}{4 \cdot 6 \cdot \cdots \cdot 2n - 1} + \frac{P}{2} \left\{ \sum_{n=1}^{\infty} \frac{1}{4 \cdot 6 \cdot \cdots \cdot 2n - 1} \right\} \right] K^{2n} \sin^{2n} \phi \right\} d\phi
\]

\[
= B \int_0^\phi \left[1 + \sum_{n=1}^{\infty} \frac{1}{4 \cdot 6 \cdot \cdots \cdot 2n - 1} + \frac{P}{2} \left\{ \sum_{n=1}^{\infty} \frac{1}{4 \cdot 6 \cdot \cdots \cdot 2n - 1} \right\} \right] K^{2n} \sin^{2n} \phi \right\} d\phi
\]

where \( B = M \alpha \), \( i \) and \( j \) are integers, \( i + j = n \) if \( n \neq 1 \), and \( i + j = 0 \) if \( n = 1 \).

The quarter-period of \( \sin^2 \phi \) is found by integrating the right member of (42) between the limits 0 and \( \frac{\pi}{2} \). Applying Wallis' formula to this integral, we get

\[
P = 4B \left[ \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{1}{4 \cdot 6 \cdot \cdots \cdot 2n - 1} + \frac{P}{2} \left[ \sum_{n=1}^{\infty} \frac{1}{4 \cdot 6 \cdot \cdots \cdot 2n - 1} \right] \right]\left[ \frac{1}{2n} \right]
\]

\[
= 2\pi B \left[ \sum_{n=1}^{\infty} \frac{1}{4 \cdot 6 \cdot \cdots \cdot 2n - 1} \left\{ \sum_{n=1}^{\infty} \frac{1}{4 \cdot 6 \cdot \cdots \cdot 2n - 1} + \frac{P}{2} \left[ \sum_{n=1}^{\infty} \frac{1}{4 \cdot 6 \cdot \cdots \cdot 2n - 1} \right] \right\} \right]
\]

(43)

The value of \( P \) in equation (43) is four times the quarter-period. By equation (34) the period of \( \phi \) is \( \frac{1}{2} P \). The right member of (43) is convergent for all values of \( K \) defined by \( 0 \leq K \leq 1 \).

**The Polar Angle.** — From the relation \( \rho' = \frac{b}{2\rho} \frac{d\phi}{dt} \) and equation
(23) we obtain
\[ \int_0^\phi d\phi = \frac{\pi}{2} \int_0^\phi \frac{d\phi}{(\rho - f(\phi))} \]
where \( f(\phi) \) stands for the bi-quadratic. Applying transformation (34) to this equation, we get
\[ \int_0^\phi d\phi = \frac{Mh}{\alpha_1^3} \int_0^\phi \left[ \frac{1}{(1 - \kappa \sin^2 \phi)} \left( \frac{m}{(1 - \kappa \sin^2 \phi)^{\frac{1}{2}}} \right) \right] d\phi, \]
where
\[ m = \frac{\alpha_2 - \alpha_1}{\alpha_3^2}, \quad \sigma = \frac{\alpha_2(\alpha_4 - \alpha_5)}{\alpha_3(\alpha_3 - \alpha_4)}. \]
Equation (44) gives the value of the polar angle for any assigned value of \( \rho \).

Expanding the right member of (44) as a power series in \( \kappa^2 \sin^2 \phi \), we have
\[ \int_0^\phi d\phi = \frac{Mh}{\alpha_1^3} \int_0^\phi \left[ \frac{1}{(1 - \kappa \sin^2 \phi)} \left( \frac{m}{(1 - \kappa \sin^2 \phi)^{\frac{1}{2}}} \right) \right] d\phi \]
\[ = \frac{Mh}{\alpha_1^3} \int_0^\phi \left[ \frac{1}{(1 - \kappa \sin^2 \phi)} \left( \frac{m}{(1 - \kappa \sin^2 \phi)^{\frac{1}{2}}} \right) \right] d\phi \]
\[ = 2\pi \int_0^\phi \left[ \frac{1}{1 + \frac{\kappa}{2} \left( \frac{2}{3} \frac{3}{5} \cdots \frac{2n-1}{2n} \right) \left( \frac{m}{(1 - \kappa \sin^2 \phi)^{\frac{1}{2}}} \right)} \right] d\phi, \]
where
\[ \sigma = \frac{(1+\epsilon)(1-x)}{\sqrt{(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_4)}}, \quad \sigma = \frac{(\alpha_2 - \alpha_3)(\alpha_4 - \alpha_5)}{\alpha_3(\alpha_3 - \alpha_4)}. \]

If now we integrate (45) by Wallis' Theorem, between the limits \( 0 \) and \( \pi \), we obtain
\[ \theta_{\alpha_4} = \pi \prod \left[ 1 + \frac{\kappa}{2} \left( \frac{2}{3} \frac{3}{5} \cdots \frac{2n-1}{2n} \right) \left( \frac{m}{(1 - \kappa \sin^2 \phi)^{\frac{1}{2}}} \right) \right] \]
\[ - \int \left[ \frac{1}{1 + \frac{\kappa}{2} \left( \frac{2}{3} \frac{3}{5} \cdots \frac{2n-1}{2n} \right) \left( \frac{m}{(1 - \kappa \sin^2 \phi)^{\frac{1}{2}}} \right)} \right] d\phi, \]
where \( \theta_{\alpha_4} \) is the value of the polar angle at the end of a half oscillation of the radius vector. This series is con-
vergent for all values of \( \kappa \) defined by \( 0 \leq k \leq 1 \), provided
\[
\frac{\alpha_3(\alpha_4-\alpha_2)}{\alpha_3(\alpha_4-\alpha_2)} < 1.
\]

2. \( \alpha^2 < \lambda \), \( \alpha_3 \) and \( \alpha_4 \) Real and Distinct, \( \alpha_1 \) and \( \alpha_2 \) conjugate imaginaries

In order to obtain the normal form in this case we will apply to (31) the transformation
\[
\phi = \frac{py+q}{y+\delta},
\]
(47)
where
\[
p = \alpha_4\beta - \alpha_3, \quad r = \alpha_4\beta + \alpha_3, \quad q = \beta - 1, \quad \delta = \beta + 1,
\]
\[
B = \frac{(\alpha_3-\alpha_2)(\alpha_4-\alpha_2)}{(\alpha_4-\alpha_3)^2}.
\]

This transformation is real and transforms the roots \( \alpha_1, \alpha_2, \alpha_3 \)
and \( \alpha_4 \) into \( -\frac{i}{\mu}, \frac{i}{\mu}, -1 \), and \( +1 \), where \( \mu \) is a real number.

Applying this transformation to equation (31), we obtain
\[
\int_0^t m^2 dt = \int_0^{2\pi} \left[ \frac{py+q}{y+\mu} \right] \left[ \frac{p^2-\mu^2}{y+\mu} \right] dy
\]
which after an elaborate reduction becomes
\[
\int_0^t m^2 dt = \frac{2\sqrt{B}}{\sqrt{B(\alpha_4-\alpha_3)(\alpha_4-\alpha_2)}} \int_0^{\pi/2} \frac{dy}{(1-y^2)(1+\mu^2 y^2)}.
\]
Reducing this further, we obtain
\[
\int_0^t m^2 dt = \mathcal{M} \int_0^{\pi/2} \left[ \frac{b}{(1-y^2)(1+\mu^2 y^2)} + \frac{c}{(1+y)(1+y)(1+\mu^2 y^2)} \right] dy,
\]
(48)
where
\[
\mathcal{M} = \frac{4B}{\sqrt{B(\alpha_4-\alpha_3)(\alpha_4-\alpha_2)}}, \quad \mu = \frac{1 + \sqrt{1 + \frac{A^2}{\lambda}}}{1 + \sqrt{1 + \frac{A^2}{\lambda}}},
\]
\[
c = \frac{r^2 - p^2}{\mu^2}, \quad A = \frac{(\alpha_3-\alpha_1)(\alpha_4-\alpha_2)}{(\alpha_3-\alpha_2)(\alpha_4-\alpha_2)},
\]
\[
d = \frac{\delta}{\mu}, \quad |y_0| \leq 1, \quad y_0 \leq y.
\]

We will now make the following quadratic substitution in (48)
\[ y = \pm \sqrt{1 - z^2}. \]  

The result is
\[
\int_0^x dt = \frac{2M}{\sqrt{i + \mu^2}} \left[ \int_{z_0}^{Z_1} \left( \frac{b}{\sqrt{(1 - z^2)(1 - \mu^2 z^2)}} + \frac{cd}{(1 - z^2)(1 - \mu^2 z^2)} \right) dz \right. \\
\left. + \int_{z_0}^{Z_2} \frac{cdz}{(1 - z^2)(1 - \mu^2 z^2)} \right].
\]

If we make the substitution \( 1 - \mu^2 z^2 = \cos \Phi \) in the last integral of the right member of this equation, we obtain for its value
\[
\frac{c e^z}{\sqrt{\mu^2 + e^2}} \tan^{-1} \left( \frac{1}{z} \sqrt{\frac{1 - \mu^2 z^2}{e^2 + \mu^2}} \right)_{Z_0}^{Z_1},
\]

where \( k^2 = \frac{\mu^2}{\mu^2 + e^2}, e^2 = \frac{1}{d^2}, |Z_1| = 1, Z_0 = Z_1. \)

Equation (50), therefore, becomes
\[
\int_0^x dt = \frac{M}{\sqrt{i + \mu^2}} \left[ \int_{z_0}^{Z_1} \left( \frac{b}{\sqrt{(1 - z^2)(1 - \mu^2 z^2)}} + \frac{cd}{(1 - z^2)(1 - \mu^2 z^2)} \right) dz \right. \\
\left. + \frac{c e^z}{\sqrt{\mu^2 + e^2}} \tan^{-1} \left( \frac{1}{z} \sqrt{\frac{1 - \mu^2 z^2}{e^2 + \mu^2}} \right)_{Z_0}^{Z_1} \right].
\]

Note that \( i \mu \) is a pure imaginary. Its two pure imaginary values are of opposite sign. One has absolute value \( > 1 \), the other absolute value \( < 1 \). Select that value for which \( 0 < \mu < 1 \). If \( \mu \) comes out negative, put \( \mu = -\mu \) and interchange notations for \( \phi \) and \( \phi' \).

If \( y \) and \( y_0 \) differ in sign \( \int_{y_0}^y \) should be split into \( \int_0^0 + \int_y^{y_0} \) and then transformed by (49), the corresponding sign before the radical of (49) being used; the negative sign with \( \int_0^0 \) and the positive sign with \( \int_y^{y_0} \). Only the negative sign of (49) was used in obtaining (51), so that both \( y \)-limits were taken negative. The use of the negative sign of (49) introduces the negative sign before the right member of (51).

Transformation (47) has been so set up that the integration of the differential equation in \( \rho \) and \( y \) must be performed over
an increasing interval; in other words the particle must be
started from $\alpha_3$, the smaller apsidal distance. If we wish to
start it from $\alpha_4$ we must either set up a different transforma-
tion which is identical in form with (34) or we must change
the origin of time by putting $t = t - \frac{p}{\omega}$ where $p$ is the period
of $\rho$ and $t$. It follows, therefore, from (47) and (49) that

$$y_o = -1, z_o = 0.$$  \hfill (52)

As $y$ starts from $-1$, $z$ may by (49) move either towards $-1$ or
$+1$. We shall, however, at the beginning of the motion inte-
grate (51) over an increasing interval. This initial relative
movement of $y$ and $z$ being known, the relative movement ever
afterwards is known, since the periods of the two are the
same. Also $\rho$ is a monotonic function of $y$.

Equation (51) gives us the value of $t$ for any selected
value of $\rho$. Thus the problem is completely solved for the case
where two of the roots are real and two are imaginary.

The Period. — We observe that the time is given by the sum
of an elementary integral and an elliptic integral of the first
kind and one of the second kind. Incidentally we may mention that
the method of reduction employed in this case is perfectly gen-
eral, and if used in case I gives an expression for the time
which is identical in form with (40). The quarter-period of
$z$ and therefore of $y$ and $\rho$ is found by integrating (51) between
the limits 0 and 1. We shall express the period $P$ as a power
series in $k^{2n}$, thus absorbing the elementary integral. To do
this write (51) in the form
\[
\int_0^t \mathcal{E} \, dt = N \int_0^1 \left\{ \frac{b}{\sqrt{(1-z^2)(1-k^2 z^2)}} + \frac{b_1}{(1+m k^2 z^2) \sqrt{(1-z^2)(1-k^2 z^2)}} \right\} \, dz,
\]

where \( b_1 = \frac{e}{d^2 + 1}, \ b_2 = \frac{e}{k^2 + 1}, \ N = \frac{M}{\sqrt{1+\mu^2}}, \ N = \frac{M}{\sqrt{1+\nu^2}}, \ m k^2 = \frac{1}{d^2 + 1}. \)

In the first two integrals of this equation put \( z = \sin \phi \), then (53) becomes
\[
\int_0^t \mathcal{E} \, dt = N \int_0^1 \left\{ \frac{b}{\sqrt{1-k^2 \sin^2 \phi}} + \frac{b_1}{(1+m k^2 \sin^2 \phi) \sqrt{1-k^2 \sin^2 \phi}} \right\} \, d\phi
\]
\[\quad + N \int_0^1 \frac{b_2 \, dz}{(1+m k^2 z^2) \sqrt{1-k^2 z^2}}. \]

(54)

We will now expand the \( \phi \)-integrals in powers of \( k^{2n} \sin^{2n} \phi \) and the \( z \)-integral in powers of \( k^{2n} z^{2n} \), we get
\[
\int_0^t \mathcal{E} \, dt = N \left[ \int_0^1 \left\{ b (1-k^2 \sin^2 \phi)^{-1} + b_1 (1+m k^2 \sin^2 \phi)^{-1} \right\} \, d\phi + \int_0^1 b_2 (1+m k^2 z^2)^{-1} \, dz \right].
\]

Simplifying, we get as a result
\[
\int_0^t \mathcal{E} \, dt = \mathcal{C}_1 \int_0^1 \left[ \sum_{n=1}^{\infty} \left\{ \frac{1}{2} \sum_{i=1}^{n-1} \frac{1}{2} \sum_{j=1}^{n-1} \frac{1}{2} \sum_{f=1}^{n-1} + \mathcal{C}_2 \left[ \frac{1}{2} \sum_{i=1}^{n-1} \frac{1}{2} \sum_{j=1}^{n-1} \frac{1}{2} \sum_{f=1}^{n-1} \right] \right\} \right] \, d\phi
\]
\[\quad + \sum_{i=1}^{\infty} \left[ \frac{1}{2} \sum_{j=1}^{n-1} \frac{1}{2} \sum_{f=1}^{n-1} \right] \, dz, \)

(55)

where
\[\mathcal{C}_1 = b + b_1, \ \mathcal{C}_2 = \frac{b_1 m}{b + b_1}, \ i + j = 0, \ i + f = 1, \ i + j = n, \ i + f \neq 1, \ i \neq 1. \]
Applying Wallis' Theorem to the $\phi$-integrals and integrating the $z$-integrals directly, we have finally

\[ P = 2\pi c_3 \left[ \left( 1 + \frac{1}{3} z \right) \sum_{n=1}^{\infty} \frac{c_4 (c_4 + c_5 b_4) + (c_4 b_4 + c_5 b_4) m \xi n \eta^{n-1}}{\left( \frac{1}{2} \xi \right)^{2j+2} + \left( \frac{1}{2} \eta \right)^{2j+2}} \right] \]

where

\[ c_3 = c_1 + b_4 \quad c_4 = \frac{1}{2} \xi \quad c_5 = \frac{1}{2} \eta \quad b_4 = \frac{2 b_5}{\pi} \]

The right member of (56) is convergent for all values of $\xi$ on the interval $0 \leq \eta \leq 1$, provided $\xi > 0.0832\ldots\ldots$
CHAPTER VII
THE ORBITS

The Classes of Orbits. — The orbits will be discussed in terms of the two parameters \( \varepsilon \) and \( \lambda \). They may be conveniently thought of as divided into two general classes, namely:

Class A: orbits that do not contain points of inflection.
Class B: orbits that do contain points of inflection.

The orbits of class A intersect the circle of radius \( \lambda \) about the origin as center, the points of intersection being inflection points since, at these points the acceleration changes sign. The two classes are separated by the orbit tangent to this circle.

The Line of Apsides. — We could express \( \rho \) as a periodic function, then the solution of the integral of areas,

\[
\Theta' = \frac{\lambda}{2\rho^2},
\]

is of the form

\[
\Theta = \mu \lambda + \varphi(\lambda),
\]

where \( \mu \) is a function of the initial conditions and \( \varphi(\lambda) \) is a power series whose coefficients have the same period as \( \rho \). In order to fix the idea, let us take the period of \( \rho \) as \( \pi \) in \( \lambda \). Then when the radius vector has completed one oscillation the value of the polar angle is

\[
\Theta = \mu \pi.
\]
It follows, therefore, that during the time $\pi$ the line of apsides turns through the angle $(\mu - 1)\pi$, the change being an advance for $\mu > 1$, and a regression for $\mu < 1$. In the problem under discussion $\mu$ is always less than unity. Hence the line of apsides always regresses. For the method of determining $\mu$ see equation (46).

**Period of the Particle.** - If $\mu$ is incommensurable with unity the particle's path will not be a closed one. The number of revolutions, $R$, made by the particle during one complete oscillation of the radius vector is given by

$$R = \frac{\mu}{2}.\$$

If the path under consideration is a closed one, the period $\overline{P}$ in $\overline{\lambda}$ is given by

$$\overline{P} = \frac{2\pi}{\mu}.$$

**The Parameters.** - Other things remaining unchanged the characteristic effect of a variation of the parameter $\lambda$ is to change the value of $\mu$; an increase in $\lambda$ produces a decrease in $\mu$, and a decrease in $\lambda$ produces an increase in $\mu$. A variation in $e$ also affects $\mu$, the two increasing or decreasing together; but the characteristic function of $e$ is to alter the loci of the apsides, upon which $\lambda$ has no effect. Other things remaining unchanged, therefore, the parameter $e$ is more effective than $\lambda$ in producing a striking change in the orbits. So in seeking to exhibit concisely the general character of the orbits, we shall place the burden of the change upon $e$.

**Diagrams of the Orbits.** - It will be well to mention here that the graphs to which we shall now refer are all drawn to the same
scale. The solid red circle is the circle of radius $\lambda$ about
the origin and is the locus of the points of inflection. The
dotted circles are the loci of the apses. In our description
we shall use $\theta_\psi$ to denote the angle described by the radius
vector as it moves from one apse to the next.

In figures I to IV we have kept the value of $\lambda$ large and
allowed $e$ to vary. For $e = 1$, the motion is along a straight
line through the origin and perpendicular to the polar axis.
As $e$ decreases, the article moves first in very narrow finger-
like loops (Fig. I). Then the apsidal circles gradually close
up on each other, shortening and flattening the loops between
them (Fig. II). Finally the points of inflection disappear
(Fig. III), and the orbit tends to become one great loop, a
circle (Fig. IV). $\theta_\psi$ does not become zero with $e$, but as $\lambda$
is kept at .9 and $e \to 0$, $\theta_\psi$ approaches 49.9° as a limiting
value.

If in figure I, $e$ is fixed while $\lambda$ varies, the elongated
loops would keep their length and general form unchanged, but
would spread farther apart or draw closer together according
as $\lambda$ decreases or increases respectively. For example if $\lambda$
is made to decrease, the points of inflection move towards
the origin, disappear when $\lambda = .1$, and the remaining orbit
then becomes of the form shown in figure VII, the loops re-
main very narrow as in figure I.

In figure V we have represented an orbit where both para-
eters are small. It does not differ radically from figure
IV. Both figures represent the transition into the circle;
but in figure IV $\theta_\psi$ is decreasing towards 49.9° as a limit,
while in figure V it is increasing towards 90° as a limit.
In figures VI and VII we have represented two intermediate types of orbits.

**Symmetry.** - Any given loop is symmetrical with respect to the radius vector connecting the origin and the farther apse of the loop. This is a property common to all orbits described under the action of a central force which is a function of the distance alone from the center.
\[ e = 0.05 \]
\[ \lambda = 0.1 \]

\[ e = 0.3 \]
\[ \lambda = 0.09 \]

Fig. V

Fig. VI

\[ e = 0.618 \]
\[ \lambda = 0.382 \]

Fig. VII