MINIMAL CURVES AND SURFACES

A THESIS

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CHAPTER I
INTRODUCTION

In pursuing the course Metric Differential Geometry, we found a particular class of curves and surfaces very interesting. In the first place their elements dealt with imaginary quantities to a large extent. Then there was the interest due to the relations between these curves and surfaces and those whose elements were real quantities. It is our purpose in this treatise to investigate some of the properties of minimal curves and surfaces.

We shall use for our treatment some preliminary ideas and theorems. These will be set forth in Chapter II. The method we shall follow will be to give a more detailed proof of the theorems and principles than was given by Eisenhart in his Differential Geometry. We shall use the notation that was used by Eisenhart in his volume mentioned above.

Also in Chapter II, we shall develop a few topics for surfaces in general to which we shall refer in the other Chapters. Some of those topics are the Dupin Indicatrix, Asymptotic lines on a Surface, Spherical Representation of a Surface, Differential Parameters of the First Order and Isothermal Orthogonal Systems.

In Chapter III we shall deal with minimal curves in space. These curves will be defined, their equation of condition will be established and their equation obtained. Then we shall consider minimal curves on a surface. Throughout we shall consider the equation of the surface in the form \( x = x(u, v), \ y = y(u, v), \ z = z(u, v) \), that is to say in the parametric
form with $u$ and $v$ as the parameters.

In Chapter IV we study minimal surfaces. The approach to the topic of minimal surfaces will be that which followed from the problem of Lagrange to find a surface of minimum area satisfying other conditions. Lines of curvature and asymptotic lines on minimal surfaces will be considered along with associate, double and algebraic minimal surfaces.

In Chapters III and IV we shall treat some exercises suggested by Eisenhart as theorems. These will be used to illustrate the general theory.

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CHAPTER II
Preliminary Theorems

1. Dupin Indicatrix. The normal curvature of a surface $S$ at the point $M$ is given by

$$
\frac{1}{R} = \frac{D\ du^2 \ E\ du^2 \ F\ dv^2 \ G\ dv^2}{E\ du^2 \ F\ dv^2 \ G\ dv^2},
$$

(1)

Theorem: (1) If the lines of curvature,

$$(ED' - FD)du^2 / (ED'' - GD)(du^2 / (FD' - GD'))dv^2 = 0$$

(2)

are parametric then (1) may be of the form

$$
\frac{1}{R} = \frac{\cos^2 \theta \ sin^2 \theta}{F},
$$

(3)

where $\theta$ is the angle between the directions whose radii of normal curvature are $R$ and $F$.

Proof: In order for (2) to be parametric

$$
\begin{align*}
ED' - FD &= 0 \\
FD'' - GD' &= 0
\end{align*}
$$

(4)

i.e., considering (4) as equations linear in $D'$ and $F$, the determinant of the coefficients is $ED'' - GD \neq 0$ except for the sphere and plane, excluding these surfaces equations (4) are satisfied if $D' = F = 0$. Conversely if $D' = F = 0$ equations (2) represent a parametric net. Consequently a necessary and sufficient condition for lines of curvature to be parametric is $D' = F = 0$. If the lines of curvature are parametric the total curvature given by

$$
K_t = \frac{1}{F_2} = \frac{D''D - D'^2}{EG - F'},
$$
becomes \( \frac{\partial^2 u}{\partial n^2} \) and the mean curvature

\[
K_m = \frac{1}{E} \cdot \frac{1}{F} = \frac{E^2 - 2FD'}{EG - F^2}
\]

reduces to

\[
\frac{ED''}{DG} = \frac{D''}{G} \cdot \frac{D}{E}
\]

that is,

\[
\frac{1}{E} = \frac{D}{G}, \quad \frac{1}{F} = \frac{D''}{D}.
\]

Therefore when the lines of curvature are parametric, let \( \rho \) and \( \rho_2 \) denote the principal radii of normal curvature of \( S \) in the directions of the curves \( v = \text{constant} \) and \( u = \text{constant} \) respectively. Let \( \phi \) be the angle between any curve on \( S \) through \( M \) and the curve \( v = \text{constant} \), then

\[
\cos \phi = \frac{1}{E} \left( E \frac{du}{ds} \right)
\]

reduces to

\[
\frac{1}{E} (Edu), \quad \text{since } F = 0 \text{ and }
\]

\[
\sin \phi = \sqrt{\frac{EG - F^2}{E}} \frac{dv}{ds}
\]

becomes

\[
\left( \frac{E}{G} \right) \frac{dv}{ds} = \sqrt{G} \frac{dv}{ds}.
\]

Therefore equation (1) becomes

\[
\frac{1}{R} = \frac{Ddu^2}{Edu^2} \cdot \frac{D' dv^2}{Gdv^2} = \frac{1}{\rho} \cos^2 \phi + \frac{1}{\rho_2} \sin^2 \phi.
\]

This is Euler's equation.

If the total curvature \( K \) is positive at a point \( M \) on \( S \), \( \rho \) and \( \rho_2 \) from (3) for the point have the same sign and \( R \) has this sign for all
directions from M. Let the tangents to the lines of curvature at M be taken for coordinate axis $\xi$ and $\eta$. Lay off segments of length $\pm \sqrt{|R|}$ from M in the two directions corresponding to R. The locus of the end-points of these segments is the ellipse

$$\frac{\xi^2}{|F_1|} - \frac{\eta^2}{|F_2|} = 1.$$  

This is the Dupin Indicatrix for the point. If $F_1 = F_2$ this indicatrix is a circle, i.e., the Dupin indicatrix at an umbilical point is a circle.

When the total curvature $K$ is negative $F_1$ and $F_2$ differ in signs, and consequently from (3) certain values of R are positive and others are negative. For the directions for which R is positive, lay off segments of length $\pm \sqrt{R}$, in the other directions, lay off segments of length $\pm \sqrt{-R}$. The locus of the end-points of these segments is the conjugate hyperbolas

$$\frac{\xi^2}{F_1} - \frac{\eta^2}{F_2} = 1,$$

and

$$\frac{\xi^2}{F_1} - \frac{\eta^2}{F_2} = -1,$$

these hyperbolas are the Dupin Indicatrix for the point.

When $K = 0$ the Dupin Indicatrix is of one of the forms

$$\frac{\xi^2}{|F_1|} = \frac{\eta^2}{|F_2|}$$

i.e., a pair of parallel lines. Therefore the point on S where the total curvature is positive is elliptic, where the total curvature is negative it is hyperbolic, and where the total curvature is zero, it is parabolic.

2. Asymptotic Lines on a Surface. By definition, when the second fundamental form for a surface is equated to zero, the equation defines an
asymptotic net on the surface, that is, the equation of an asymptotic net is

\[ Ddu^2 + 2D'dudv + D''dv^2 = 0. \quad (5) \]

Theorem 1. A necessary and sufficient condition that the asymptotic lines (1) upon \( S \) be parametric is that

\[ D = D'' = 0. \]

Proof: For if \( D = D'' = 0 \), then (5) becomes

\[ 2D'dudv = 0 \]

i.e., \( du = 0 \), whence \( u \) is a constant, and \( dv = 0 \), whence \( v \) is a constant. Hence the condition is necessary. Conversely if the asymptotic lines are parametric, i.e., \( u = \text{constant} \) and \( v = \text{constant} \), then \( du = 0 \) and \( dv = 0 \).

Therefore, \( \varphi (u,v) \) \( dudv = 0 \), defines a net upon \( S \), where \( \varphi (u,v) \) may be chosen to be \( 2D' \). This net is identical with (5) the asymptotic net if

\[ D = D'' = 0. \]

Therefore the condition is sufficient.

Theorem 2. A necessary and sufficient condition that the asymptotic lines upon a surface \( S \) form an orthogonal system is that the mean curvature \( K \) of the surface be zero.

Proof: From theorem 1, if the asymptotic lines are parametric

\[ D = D'' = 0. \]

Now if \( \alpha, \beta, \gamma, \alpha, \beta, \gamma \) are the direction - cosines of the tangents to the parametric curve at their common point, then the equations of surface \( S \) are

\[ x = x(u,v), \quad y = y(u,v), \quad z = z(u,v). \]

\[ \alpha_v = \frac{\partial x}{\partial u}, \quad \beta_v = \frac{\partial y}{\partial u}, \quad \gamma_v = \frac{\partial z}{\partial u}, \quad \alpha_u = \frac{\partial x}{\partial v}, \quad \beta_u = \frac{\partial y}{\partial v}, \quad \gamma_u = \frac{\partial z}{\partial v}. \]

Let \( \omega \) be the angle between the parametric curve at their common point.
Then
\[ \cos \omega = a_\omega a_\nu + b_\nu b_\nu + c_\nu c_\nu = \frac{F}{EG}. \]

If these curves are orthogonal,
\[ F = 0, \]
for, \( \cos 90^\circ = 0 \)
and conversely if \( F = 0 \) the curves are orthogonal. Therefore a necessary
and sufficient condition for the parametric curves to be orthogonal is
\[ F = 0. \]

From the relation for the mean curvature \( K_m \) of the surface at a point
\[ \frac{1}{f_1} \neq \frac{1}{f_2} = \frac{ED'' - GD' - 2FD'}{H^2}, \]
if the asymptotic lines upon the surfaces form an orthogonal system, then
\[ D'' = E = F = 0 \]
and hence the mean curvature \( K_m = 0 \).

3. Spherical Representation of a Surface. Consider a surface \( S \) and
a sphere of unit radius with center at the origin of coordinate system.
Draw radii of the parallel to the positive directions of the normals to
\( S \). Take the extremities of these radii as spherical images of the corresponding points on \( S \). As a point \( M \) moves along a curve on \( S \), its image \( M \) describes a curve on the sphere. Let us consider a portion of the surface, where no two normals are parallel. Then the portions of the surface and sphere will be in a one-to-one correspondence. This map of the surface \( S \) upon the sphere is called the spherical representation of the surface, or the Gaussian Representation. The coordinates of \( M \) are the direction cosines of the normals to the surface, namely \( X, Y, Z \). For let the coordinates of \( M \) be \( x, y, z \). Then since the radius of the sphere to \( M \) is parallel to the normal to \( S \) at \( M \), it follows that
\[ X = \frac{x}{1}, \quad Y = \frac{y}{1}, \quad Z = \frac{z}{1}. \]

Let the fundamental coefficients of the spherical representation of \( S \) be denoted by \( \varepsilon, \varphi, \gamma \). Then

\[ \varepsilon = \sum \left( \frac{\partial X}{\partial \mu} \right)^2, \quad \varphi = \sum \frac{\partial Y}{\partial \mu} \frac{\partial Y}{\partial \nu}, \quad \gamma = \sum \left( \frac{\partial Z}{\partial \nu} \right)^2. \] (6)

Then the square of the linear element of the spherical representation of \( S \) is

\[ d\sigma^2 = \varepsilon \, d\mu^2 + 2 \, \varphi \, d\mu \, d\nu + \gamma \, d\nu^2. \] (7)

For the surface \( S \) it has been shown that

\[ \frac{\partial Y}{\partial \mu} \frac{F D' - G D}{H^2} \neq \frac{F D - E D'}{H^2} \frac{\partial X}{\partial \nu}, \]

\[ \frac{\partial Y}{\partial \nu} \frac{F D'' - G D}{H^2} \neq \frac{F D' - E D''}{H^2} \frac{\partial X}{\partial \mu}, \]

\[ \frac{\partial Y}{\partial \mu} \frac{F D' - G D}{H^2} \neq \frac{F D - E D'}{H^2} \frac{\partial Y}{\partial \nu}, \]

\[ \frac{\partial Y}{\partial \nu} \frac{F D'' - G D}{H^2} \neq \frac{F D' - E D''}{H^2} \frac{\partial Y}{\partial \mu}, \]

\[ \frac{\partial Z}{\partial \mu} \frac{F D' - G D}{H^2} \neq \frac{F D - E D'}{H^2} \frac{\partial Z}{\partial \nu}, \]

\[ \frac{\partial Z}{\partial \nu} \frac{F D'' - G D}{H^2} \neq \frac{F D' - E D''}{H^2} \frac{\partial Z}{\partial \mu}. \] (8)

By means of equations (8) equations (6) can be given the forms

\[ \varepsilon = \frac{1}{H^2} \left( (G D')^2 - 2 F D' D' E D' \right) \]

\[ \varphi = \frac{1}{H^2} \left( (G D'') - F (D D') D' \right) - E D' D'' \] (9)

\[ \gamma = \frac{1}{H^2} \left( (G D'')^2 - 2 F D' D'' E D'' \right) \]
Proof: For \( \Sigma \left( \frac{2\chi}{2\nu} \right)^2 = \Sigma \left[ \left( \frac{FD' - GD}{H^2} \right)^2 \left( \frac{2\chi}{2\nu} \right)^2 \right] \),

\[
2 \left( \frac{FD' - GD}{H^2} \right)^2 \frac{FD' - ED'}{H^2} \frac{2\chi}{2\nu} \frac{2\nu}{2\chi}
\]

\[
f \left( \frac{FD - ED'}{H^2} \right)^2 \left( \frac{2\chi}{2\nu} \right)^2
\]

\[
= \frac{(FD' - GD) (FD' - GD) E (FD - ED') F}{H^4} \frac{(FD - ED') (FD' - GD) G (FD' - GD) E}{H^4}
\]

\[
= \frac{(EG - F^2) \left[ (-D(FD' - GD) - D' (FD - ED)) \right]}{(EG - F^2)^2}
\]

and

\[
\Sigma \frac{2\chi}{2\nu} = \Sigma \left[ \left( \frac{FD' - GD}{H^2} \right)^2 \left( \frac{2\chi}{2\nu} \right)^2 \right]
\]

\[
\]

\[
= \frac{(FD' - GD) (FD' - GD) E (FD' - ED') (FD' - ED') (FD' - GD) G (FD' - GD) E}{H^4}
\]

\[
= \frac{(EG - F^2) \left[ GDD + F(FD'' \not\equiv D'^2) \not\equiv ED'D'' \right]}{H^4}
\]

Likewise

\[
\Sigma \left( \frac{2\chi}{2\nu} \right)^2 = \Sigma \left[ \left( \frac{FD'' - GD'}{H^2} \right)^2 \left( \frac{2\chi}{2\nu} \right)^2 \right]
\]

\[
= \frac{(FD'' - GD') E (FD - ED') (FD'' - ED') (FD' - GD') G (FD'' - ED') (FD' - GD') G}{H^4}
\]

\[
= \frac{(E_D'^2 - 2FD''GD') F (E'' - FGD) G (E'' - FGD) G}{H^4}
\]

\[
= \frac{EG^2 D' \not\equiv 2FD''ED'' \not\equiv FGD'2 \not\equiv FGD''2}{H^4}
\]
\[
\frac{E^2 (E - F^2)}{H^2} \neq \frac{G^2 (E - F^2)}{H^2} - 2F^2 \frac{E^2}{H^2}
\]

We may express \( \epsilon, \tau, \) and \( \phi \) in terms of mean and total curvature as follows:

\[
\epsilon = K_m D - KE, \quad \tau = K_m D' - KF, \quad \phi = K_m D'' - KG.
\]  

(10)

Proof: \( K = \frac{1}{F' P^2} \frac{D^2 - D' ^2}{H^2} \)

and

\[
K_m = \frac{1}{F'} \frac{1}{P^2} \frac{E D'' - G D - 2F D'}{H^2}.
\]

From these equations we can show that

\[
K_m D - KE = \frac{GD^2 - 2FDD' \neq E D''}{H^2} = \frac{GD^2 - 2FDD' \neq E D''}{H^2}.
\]

Hence by (9)

\[
\epsilon = K_m D - KE.
\]

Also

\[
K_m D' - KF = \frac{ED'D'' \neq GDD' - 2F' D' - FDD'' \neq FD'}{H^2} = \frac{GDD' - F(DD'' \neq D') \neq ED'D''}{H^2}.
\]

Therefore,

\[
\tau = K_m D' - KF.
\]

Similarly

\[
K_m D'' - KG = \frac{E D''^2 - 2F D'D'' \neq G D'^2}{H^2} = \frac{1}{H^2} \left[ GD'^2 - 2F D'D'' \neq E D''^2 \right].
\]
Therefore
\[ y = K_m D'' - KG. \]
Let us next compute the linear element for the spherical representation.

Substituting equations (10) in (2), we have
\[
d\sigma^2 = (K_m D - KE)du^2 + (K_m D' - KP) 2du dv + (K_m D'' - KG)dv^2
= K_m(Du^2 + 2D'du dv + D''dv^2) - KE^2 - 2KD'du dv - Gdv^2
= [Edu^2 + 2Fdu dv + Gdv^2] \left[ K_m \frac{Du^2 + 2D'du dv + D''dv^2 - KE^2}{Edu + 2Fdu dv + Gdv^2} \right]. \tag{11}
\]

But the fraction in the second bracket is normal curvature of the surface at a point and is represented by \( \frac{1}{R} \). Therefore (11) may be written
\[
d\sigma^2 = \left( \frac{K_m - K}{R} \right) ds^2. \tag{12}
\]

From (10)
\[
H = \sqrt{\varepsilon} \frac{\sqrt{H^2 - D^2}}{E} = \sqrt{\left[ K_m D - KE \right] (K_m D'' - KG) - (K_m D' - KP)^2}
= \sqrt{K_m^2D'' - K_m KD - K_m KED'' - K^2EG - K_m^2D' - 2K_m PD' - K^2F^2}
= \sqrt{K^2(DG - F^2 - K_m^2(D'' - D'^2) - K_m^2(D'D'' - G)}
= \sqrt{K^2H^2 - K_m^2H^2} = \sqrt{K^2H^2 - \varepsilon KH},
\]
where \( \varepsilon \) is \( \pm 1 \) according as total curvature is positive or negative.

Since equations (9) are linear in \( E, F \) and \( G \), we may solve for these quantities and get
\[
E = \frac{1}{H^2} \left[ D^2 - 2FDD' \right],
F = \frac{1}{H^2} \left[ DD' - D'' \right],
G = \frac{1}{H^2} \left[ D'^2 - 2FDD'' \right]. \tag{13}
\]

Proof: (9) may be written in form
\[
GD^2 - 2FDD' \neq ED'^2 = H^2 \varepsilon,
\]
Solving these equations for $G$, $F$ and $E$ we get

\[
G = \frac{H^2 \varepsilon - 2DD^1}{H^2 \varepsilon - 2DD^1} \frac{D^1 \varepsilon}{D^1 \varepsilon - 2DD^1} = \frac{D^1 \varepsilon}{D^1 \varepsilon - 2DD^1}
\]

\[
E = \frac{2DD^1}{2DD^1 - HD^1 \varepsilon} \frac{D^1 \varepsilon}{D^1 \varepsilon - 2DD^1} = \frac{2DD^1}{2DD^1 - HD^1 \varepsilon}
\]

\[
F = \frac{2DD^1}{2DD^1 - HD^1 \varepsilon} \frac{D^1 \varepsilon}{D^1 \varepsilon - 2DD^1} = \frac{2DD^1}{2DD^1 - HD^1 \varepsilon}
\]
Let us compute now the lines of curvature. The normals to the surface along the lines of curvature form a developable surface. Let \( x, y, z \) be the coordinates of a point \( M \) on the normal to the surface at a point \( M(x, y, z) \). Then

\[
\begin{align*}
x_i &= x \not/ rX, & y_i &= y \not/ rY \text{ and } z_i &= z \not/ rZ,
\end{align*}
\]

where \( r \) is the distance \( MM \), and \( X, Y, Z \) are the direction cosines of the normal. For a displacement in the direction of a line of curvature, we have

\[
\frac{\partial x}{\partial u} \not/ \frac{\partial y}{\partial v} \not/ \frac{\partial z}{\partial w} = 0,
\]

\[
\frac{\partial y}{\partial u} \not/ \frac{\partial z}{\partial v} \not/ \frac{\partial x}{\partial w} = 0,
\]

\[
\frac{\partial z}{\partial u} \not/ \frac{\partial x}{\partial v} \not/ \frac{\partial y}{\partial w} = 0.
\]

Multiply equations (14) by \( \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial w} \) respectively, and get

\[
\frac{\partial x}{\partial u} \not/ \frac{\partial y}{\partial v} \not/ \frac{\partial z}{\partial w} = 0,
\]

\[
\frac{\partial y}{\partial u} \not/ \frac{\partial z}{\partial v} \not/ \frac{\partial x}{\partial w} = 0,
\]

\[
\frac{\partial z}{\partial u} \not/ \frac{\partial x}{\partial v} \not/ \frac{\partial y}{\partial w} = 0.
\]

Add and get

\[
-Ddu - D'ev \not/ r[\varepsilon \not/ du \not/ \varepsilon dv] = 0,
\]

or,

\[
Ddu \not/ D'ev - r[\varepsilon \not/ du \not/ \varepsilon dv] = 0.
\]

Likewise multiply (14) by \( \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial w} \) respectively, and get

\[
\frac{\partial x}{\partial u} \not/ \frac{\partial y}{\partial v} \not/ \frac{\partial z}{\partial w} = 0,
\]

\[
\frac{\partial y}{\partial u} \not/ \frac{\partial z}{\partial v} \not/ \frac{\partial x}{\partial w} = 0,
\]

\[
\frac{\partial z}{\partial u} \not/ \frac{\partial x}{\partial v} \not/ \frac{\partial y}{\partial w} = 0.
\]
add and get

$$-D'du - D''dv \neq r \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} du \neq \int \left( \frac{\partial^2 f}{\partial x^2} \right) dv = 0,$$

or,

$$D' du \neq D'' dv - \left[ r \frac{\partial f}{\partial x} \neq \int \left( \frac{\partial f}{\partial y} \right) dv \right] = 0. \tag{16}$$

Eliminating $r$ between (15) and (16) we obtain, by solving (15) for $r$ and substituting this value in (16)

$$(D - r e) du \neq (D - r f) dv = 0.$$

Equation (17) is the lines of curvature of the spherical representation.

Now let us eliminate $du$ and $dv$ from (15) and (16), which may be written in the forms

$$(D - r e) du \neq (D - r f) dv = 0,$$

and

$$(D' - r f) du \neq (D'' - r g) dv = 0,$$

multiply the first of these equations by $(D' - r f)$ and the second by $(D - r e)$ and eliminate $du$ and get

$$\left[(D' - r f)^2 - (D - r e)(D'' - r g)\right] dv = 0.$$

Therefore

$$D' + 2rD' f \neq r^2 f'^2 - DD'' \neq f f'^2 \neq r e D'' - r^2 g = 0$$

or,

$$(\delta y - \delta x^2) r^2 - (\delta D'' \neq \delta D - 2 D' f) r \neq DD'' - D' = 0. \tag{18}$$

Equation (18) gives the principal radii for the spherical representation of the surface. If $r_1$ and $r_2$ are the principal radii given by (18), from Theory of Equations we have

$$r_1 \neq r_2 = \frac{\delta D'' \neq \delta D - 2 D' f}{\delta x^2} \neq$$

and

$$r_1, r_2 = \frac{DD'' - D' = 2}{\delta x^2}. \tag{19}$$
By means of (19) we may write (13) as

\[ E = (\rho_1 \rho_2) D - \rho_1 \rho_2 E, \]

\[ F = (\rho_1 \rho_2) D^1 - \rho_1 \rho_2 F, \quad (20) \]

\[ G = (\rho_1 \rho_2) D^2 - \rho_1 \rho_2 G. \]

Proof: 

\[
(\rho_1 + \rho_2) D - \rho_1 \rho_2 E = \frac{E^D D^2 - \frac{\gamma}{D^1 D^2} - 2 \frac{\gamma D^2}{D^1 D^2} - \frac{\varepsilon}{D^1 2}}{\mu^2}
\]

\[ = \frac{1}{\mu^2} \left[ E^D D^2 - 2 \frac{\gamma D^2}{D^1 D^2} - \frac{\varepsilon}{D^1 2} \right]. \]

Therefore

\[ E = (\rho_1 + \rho_2) D - \rho_1 \rho_2 E, \]

also

\[
(\rho_1 + \rho_2) D^1 - \rho_1 \rho_2 F = \frac{E D^1 D^2 - \frac{\gamma D^2}{D^1 D^2} - 2 \frac{\gamma D^2}{D^1 D^2} - \frac{\varepsilon D^1 2}{D^1 2}}{\mu^2}
\]

\[ = \frac{\gamma D^2}{D^1 D^2} - \frac{\varepsilon D^1 2}{D^1 2} \]

Therefore

\[ F = (\rho_1 + \rho_2) D^1 - \rho_1 \rho_2 F. \]

Similarly

\[
(\rho_1 + \rho_2) D^2 - \rho_1 \rho_2 G = \frac{E D^2 D^2 - \frac{\gamma D^2}{D^1 D^2} - 2 \frac{\gamma D^2}{D^1 D^2} - \frac{\varepsilon D^1 2}{D^1 2}}{\mu^2}
\]

\[ = \frac{\gamma D^1 2}{D^1 D^2} - 2 \frac{\gamma D^1 D^2}{D^1 D^2} - \frac{\varepsilon}{D^1 2} \]

Therefore

\[ G = (\rho_1 + \rho_2) D^2 - \rho_1 \rho_2 G. \]

4. Conformal Representation of Two Surfaces. When a one-to-one correspondence of any kind is established between the points of two surfaces, either surface may be said to be represented on the other. However, if it is possible to represent one surface upon another in such a way that the angles between corresponding lines on the surfaces are equal, we say
that one surface has conformal representation on the other.

Let us suppose that the 2 surfaces $S$ and $S'$ are referred to a corresponding system of real lines in terms of the same parameters $u$ and $v$, and that the corresponding points have the same curvilinear coordinates. Then their linear elements can be put in the following forms

$$ds^2 = Edu^2 / 2Fdudv / Gdv^2$$

and

$$ds'^2 = E'du^2 / 2F'dudv / G'dv^2.$$ 

Since the angles $\omega$ and $\omega'$ between the coordinate lines at corresponding points must be equal it is necessary that

$$\frac{F}{\sqrt{EG}} = \frac{F'}{\sqrt{E'G'}} \quad (21)$$

Let $\omega$ and $\omega'$ denote the angles in which a curve on $S$ and the corresponding curve on $S'$ respectively make with the curve $v = \text{constant}$ at points of the former curves. Then from the formula for the angles between curves, we have,

$$\sin \omega = \frac{H}{E} \frac{dv}{ds}, \quad \sin (\omega - \omega) = \frac{H}{\sqrt{G}} \frac{du}{ds},$$

$$\sin \omega' = \frac{H'}{E'} \frac{dv}{ds'}, \quad \sin (\omega' - \omega') = \frac{H'}{\sqrt{G'}} \frac{du}{ds'}.$$ 

By hypothesis

$$\omega' = \pm \omega, \quad \omega' = \pm \omega,$$

according as the angles have the same or opposite sense. Hence we have,

$$\frac{H'}{\sqrt{E'}} \frac{dv}{ds'} = \frac{H}{E} \frac{dv}{ds}$$

and

$$\frac{H'}{\sqrt{G'}} \frac{du}{ds'} = \frac{H}{G} \frac{du}{ds},$$

according to the sense of the angles. From these equations we find by division that
By combining this last equation with (21) we get

\[
\frac{\sqrt{E}}{\sqrt{G}} = \frac{\sqrt{E'}}{\sqrt{G'}}.
\]

where \( t^2 \) is a factor of proportionality and is a function of \( u \) and \( v \) in general.

From (22) we have,

\[
ds^{'2} = E'du^2 / F'dudv / G'dv^2 = (Edu^2 / Fдуdv / Gdv^2 )t^2
\]

or,

\[
ds^{'2} = t^2ds^2 . \quad (23)
\]

Thus we have proved the following:

**Theorem 3.** A necessary and sufficient condition that the representation of two surfaces referred to a corresponding system of lines be conformal is that the first fundamental coefficients of the two surfaces be proportional, the factor of proportionality being a function of the parameter.

From (22) we see that when two surfaces \( S \) and \( S' \) have conformal representation the minimal curves upon \( S \) and \( S' \) correspond. Conversely when two surfaces \( S \) and \( S' \) are referred to a corresponding system of lines; if the minimal lines on \( S \) and \( S' \) correspond, (22) must hold.

Hence

**Theorem 4.** A necessary and sufficient condition that the representation of two surfaces upon one another to be conformal is that the minimal lines correspond.

Let the minimal lines upon the two surfaces be known, and taken as parametric. Then the linear elements of \( S \) and \( S' \) are of the form
\[ ds^2 = \lambda \, d\alpha \, d\beta \]

and
\[ ds'^2 = \lambda \, d\alpha \, d\beta \]

Therefore a conformal representation is defined in the most general way by the equations
\[ \alpha = F(\alpha), \quad \beta = F(\beta) \quad (25) \]
or
\[ \alpha = F(\beta), \quad \beta = F(\alpha) \quad (26) \]

where \( F \) and \( F' \) are arbitrary functions which must be conjugate imaginaries when \( S \) and \( S' \) are real.

5. Differential Parameters of the First Order. Let \( S \) be a surface referred to any system of coordinates \( u \) and \( v \) and let \( \phi(u,v) \) be a function of \( u \) and \( v \). When the values of the coordinates of a point \( M \) are substituted in \( \phi \), we obtain a number \( c \) and consequently the curve
\[ \phi(u,v) = c \quad (27) \]

passes through \( M \). In a displacement from \( M \) along this curve the value of \( \phi \) remains the same, but in any other directions the value of \( \phi \) changes and the rate of change is given by
\[
\frac{d\phi}{ds} = \frac{\partial \phi}{\partial u} \frac{du}{ds} + \frac{\partial \phi}{\partial v} \frac{dv}{ds} = \frac{\partial \phi}{\partial u} \frac{du}{ds} + \frac{\partial \phi}{\partial v} \frac{dv}{ds}
\]

\[
= \frac{\partial \phi}{\partial u} \frac{du}{dv} + \frac{\partial \phi}{\partial v} \frac{dv}{du} = \frac{\partial \phi}{\partial u} + \frac{\partial \phi}{\partial v} K
\]

where \( K = \frac{dv}{du} \) and determines the direction from the point \( M \).

Let us consider \( \left| \frac{d\phi}{ds} \right| \) and write
\[ A = \left| \frac{d\varphi}{ds} \right| = e \left[ \frac{\frac{\partial T}{\partial x} + \frac{\partial T}{\partial y} k}{\sqrt{(E f + G k)}} \right] \]  

(28)

where \( e \) is \( \pm 1 \) according as the sign of the numerator is positive or negative, for the sign of the denominator is considered positive. The minimal value of \( A \) is zero and corresponds to the direction along the curve (27).

Let us find the maximum value of \( A \) by equating to zero the \( \frac{dA}{dk} \), i.e.,

\[(E \frac{\partial T}{\partial y} - F \frac{\partial T}{\partial x}) / (F \frac{\partial T}{\partial y} - G \frac{\partial T}{\partial x})K = 0. \]  

(29)

(29) gives the critical value for \( A \) to be a maximum. The value of \( K \) given by (29) determines a direction at right angles to the tangent to the curve \( \phi = c \) at the point, for the condition that a family of curves

\[ \phi (u, v) = 0 \]

admit of a family of orthogonal trajectories is that (29) be satisfied.

Solve (29) for \( K \) and substitute in (28) and get

\[ \left| \frac{d\varphi}{ds} \right| = \frac{\sqrt{E (\frac{\partial T}{\partial x})^2 - 2F \frac{\partial T}{\partial x} \frac{\partial T}{\partial y} + G (\frac{\partial T}{\partial x})^2}}{\sqrt{EG - F^2}} \]  

(30)

If we put

\[ \Delta \varphi = \frac{E (\frac{\partial T}{\partial y})^2 - 2F \frac{\partial T}{\partial x} \frac{\partial T}{\partial y} + G (\frac{\partial T}{\partial x})^2}{EG - F^2} \]  

(31)

we may write (30) as

\[ \left( \frac{d\varphi}{ds} \right)^2 = \Delta \varphi \]  

(32)

where the differential quotient \( \frac{d\varphi}{ds} \) corresponds to the direction normal to the curve \( \phi = \text{constant} \). \( \Delta \varphi \) is called the differential parameter of the first order. From equation (32) the left hand member is evidently independent of the nature of the parameters \( u \) and \( v \) to which the surface is
referred. Therefore the right hand member is independent of these parameters, i.e., \( \Delta_1 \varphi \) is an invariant with reference to the parameters of the surface.

Let \( \varphi = \text{constant} \) and \( \psi = \text{constant} \) be the equations of two curves upon a surface through a point \( M \) and let \( \theta \) denote the angle between the tangents at \( M \). If we put

\[
\Delta_1(\varphi, \psi) = E \frac{\partial \varphi}{\partial v} \frac{\partial \psi}{\partial u} - F \left( \frac{\partial \varphi}{\partial v} \frac{\partial \psi}{\partial u} + \frac{\partial \varphi}{\partial u} \frac{\partial \psi}{\partial v} \right) + G \frac{\partial \varphi}{\partial u} \frac{\partial \psi}{\partial v}, \quad (33)
\]

\[
EG = F^2
\]

\[
\cos \theta = \frac{\Delta_1(\varphi, \psi)}{\sqrt{\Delta_1 \varphi \Delta_1 \psi}}. \quad (34)
\]

Now \( \cos \theta \) is an invariant for transformation of coordinates. Therefore from (34) \( \Delta_1(\varphi, \psi) \) is an invariant, since we saw that \( \Delta_1 \varphi \) and \( \Delta_1 \psi \) are invariant. This invariant, \( \Delta_1(\varphi, \psi) \) is a mixed differential parameter of the first order. It follows at once from (34) that

\[
\Delta_1(\varphi, \psi) = 0
\]

is the condition of orthogonality of the curves \( \varphi = \text{constant} \) and \( \psi = \text{constant} \).

Also \( \sin \theta \) can be written in the following form:

\[
\sin \theta = \frac{1}{\sqrt{\Delta_1 \varphi \Delta_1 \psi}} \left( \frac{\partial \varphi}{\partial u} \frac{\partial \psi}{\partial v} - \frac{\partial \varphi}{\partial v} \frac{\partial \psi}{\partial u} \right)
\]

or we may write \( \sin \theta \) as

\[
\sin \theta = \frac{\Theta(\varphi, \psi)}{\sqrt{\Delta_1 \varphi \Delta_1 \psi}}, \quad (35)
\]

where

\[
\Theta(\varphi, \psi) = \frac{1}{H \left( \frac{\partial \varphi}{\partial u} \frac{\partial \psi}{\partial v} - \frac{\partial \varphi}{\partial v} \frac{\partial \psi}{\partial u} \right)}. \quad (36)
\]

Since all the functions in (35) except \( \Theta(\varphi, \psi) \) are known to be invariants under transformation of coordinates, it is also an invariant; it is a mixed differential parameter of the first order. From (34) and (35) it follows that

\[
\Delta_1^2(\varphi, \psi) + \Theta^2(\varphi, \psi) = \Delta_1 \varphi \cdot \Delta_1 \psi, \quad (37)
\]
Consequently the three invariants $A, \varphi, A(\varphi, \psi)$ and $\Theta(\varphi, \psi)$ are not independent of each other.

Proof of (37). Squaring (34) and (35) and adding we get
\[
\cos^2 \varepsilon \neq \sin^2 \varepsilon = \frac{A^2(\varphi, \psi) + \Theta^2(\varphi, \psi)}{A, \varphi, A, \psi}.
\]
\[A^2(\varphi, \psi) + \Theta^2(\varphi, \psi) = A, \varphi, A, \psi.\]

From (32) and (33) it follows that
\[\Delta_t u = \frac{G}{H^2}, \Delta_t (u, v) = -\frac{F}{H^2}, \Delta_v v = \frac{E}{H^2},\]
and from these we find that
\[\Theta^2(u, v) = \Delta_t u \cdot A, \psi - \Delta^2_t (u, v) = \frac{1}{H^2}.\quad (37)\]

Consequently
\[E = \frac{\Delta_t v}{\Theta^2(u, v)}, \quad F = -\frac{\Delta_t (u, v)}{\Theta^2(u, v)}, \quad G = \frac{\Delta_t u}{\Theta^2(u, v)}.\]

6. Isothermic Orthogonal Systems. When $\alpha$ and $\beta$ are symmetric co-
ordinates of a surface the linear element may be put into the form
\[ds^2 = \lambda d\alpha d\beta \quad (38)\]
where $\lambda$ is a function of $\alpha$ and $\beta$, i.e., the parametric curves on the
surface are minimal.

The general linear element of a surface can be written as
\[ds^2 = (\sqrt{E} du / \sqrt{E} dv)(\sqrt{F} du / \sqrt{F} dv).\quad (39)\]

When the surface is real and the coordinates also the factors in (39) are
conjugate imaginaries. Let $t$ and $t'$ denote the integrating factors of the
respective factors of the right member of (39). Then a pair of symmetric
coordinates of the surface is given by the quadrature
\[t(\sqrt{E} du / \sqrt{E} dv) = d\alpha, \quad (40)\]
\[t' (\sqrt{F} du / \sqrt{F} dv) = d\beta', \quad (40)\]
t may be taken as the conjugate imaginary of t. In this case \( \alpha \) and \( \beta \) are conjugate imaginaries also.

Let
\[
\alpha = \varphi - i \psi \quad \text{and} \quad \beta = \varphi + i \psi.
\]

If these values are substituted in equation (38) we get
\[
ds^2 = \lambda \left[ d\varphi / i \psi \right] \left[ d\varphi - id\psi \right] = \lambda \left( d\varphi^2 - d\psi^2 \right). (41)
\]

Since \( \varphi \) and \( \psi \) are the parameters for the surface, we see from (41) that the curves \( \varphi = \) constant and \( \psi = \) constant, i.e., the parametric curves form an orthogonal system. Also the elements of arc of these parametric curves are \( \nabla \varphi d\varphi \) and \( \nabla \psi d\psi \) respectively. Consequently when the increments \( d\varphi \) and \( d\psi \) are taken equal, the four points \( (\varphi, \psi), (\varphi + d\varphi, \psi), (\varphi, \psi + d\psi), (\varphi + d\varphi, \psi + d\psi) \) are the vertices of a small square. Hence the parametric curves \( \varphi = \) constant and \( \psi = \) constant, divide the surface into a network of small squares. For this reason these parametric curves are called isometric curves and \( \varphi \) and \( \psi \) are called isometric parameters.

If the linear element of a surface is given in the form (41) and the parameters are changed in accordance with the equations,
\[
\varphi = f_1(u) \quad \text{and} \quad \psi = f_2(v)
\]
the linear element becomes
\[
ds^2 = \lambda \left( f_1'^2 du^2 - f_2'^2 dv^2 \right), (42)
\]
where the accents denote differentiation.

Proof: In \( ds^2 = \lambda \left( d\varphi^2 - d\psi^2 \right) \)
substitute
\[
d\varphi = f_1'(u) du
\]
and
\[ d\psi = f^i_2(v) \, dv \]
and get
\[ ds^2 = \lambda(f^i_2(u) \, du^2 + f^i_2(v) \, dv^2) \]
This proves (42).

This transformation of the parameters does not change the coordinate lines, but the coefficients are now no longer equal, but
\[ \frac{E}{G} = \frac{U}{V}, \quad (43) \]
where \( U \) and \( V \) are functions of \( u \) and \( v \) respectively. Conversely when (43) is satisfied the linear element may be written
\[ ds^2 = \lambda(Udu^2 + Vdv^2) \]
and by the transformation of coordinates
\[ \phi = \int U \, du \quad (44) \]
and
\[ \psi = \int V \, dv \]
the linear element may be written in the form (41), namely
\[ ds^2 = \lambda(d\phi^2 + d\psi^2). \]

**Proof:** \( ds^2 = \lambda(Udu^2 + Vdv^2). \)

From (44) we get
\[ d\phi = \sqrt{U} \, du \]
and
\[ d\psi = \sqrt{V} \, dv. \]

Whence
\[ du = \frac{d\phi}{\sqrt{U}} \]
and
\[ dv = \frac{d\psi}{\sqrt{V}}. \]
Then

\[ ds^2 = \lambda \left( d\varphi^2 + d\psi^2 \right). \]

Hence we have the following:

**Theorem 4.** A necessary and sufficient condition that an orthogonal system on a surface form an isotheral system, is that the coefficients of the corresponding linear element satisfy a relation of the form (43).
CHAPTER III
MINIMAL CURVES

1. Minimal Curves in Space. The linear element of space curve may be written in the form
\[ ds^2 = dx^2/\, dy^2/\, dz^2. \]
If this is equal zero, i.e.,
\[ dx^2/\, dy^2/\, dz^2 = 0, \]
the curve defined by the point \( M(x, y, z) \) is called a minimal curve. It is obvious from definition that such curves are imaginary, for unless \( dx, dy \) and \( dz \) are all zero, in which case the curve reduces to a point at least one of the coordinates \( x, y \) or \( z \) must be imaginary. We may therefore define minimal curves as curves of zero length.

Equation of condition. Let a curve \( C \) be defined by the equations
\[ x = f_1(u), \quad y = f_2(u), \quad z = f_3(u), \]
where these functions satisfy the condition
\[ f_1^2/\, f_2^2/\, f_3^2 = 0 \quad (1) \]
that is
\[ ds^2 = dx^2/\, dy^2/\, dz^2 = 0, \]
where the primes denote differentiation with respect to \( u \).

(1) may be written
\[ f_1^2 / f_2^2 = - f_3^2 \]
or,
\[ (f_1^2/\, if_2^2)(f_1^2 - if_2^2) = - f_3^2. \]
By the theorem of proportion the last equation may be written

\[
\frac{f'_1 - if'_3}{-f'_3} = \frac{f'_2}{f'_1 - if'_2} = v, \tag{2}
\]

where \(v\) is a constant or a function of \(u\). Considering (2) as two equations in \(f'_1, f'_2,\) and \(f'_3\) and solving for \(f'_1\) and \(f'_2\) in terms of \(f'_3\) we get

\[
f'_1 = \frac{-f'v + i}{f'_3 - iv} = \frac{f'_3 (1-v^2)}{2v}
\]

and

\[
f'_2 = \frac{1 - f'v}{f'_3 - iv} = \frac{f''_3 (1-v^2)}{-2iv}.
\]

From above it follows that

\[
f'_1 : f'_2 = \frac{1-v^2}{2} : \frac{i(1-v^2)}{2}.
\]

also

\[
f'_1 : f'_3 = \frac{1-v^2}{2} : v
\]

and

\[
f'_2 : f'_3 = \frac{i (1-v^2)}{2} : v.
\]

Therefore

\[
f'_1 : f'_2 : f'_3 = \frac{1-v^2}{2} : i \frac{(1-v^2)}{2} : v, \tag{3}
\]

Let the common ratio in (3) be a function of \(u\), say \(f(u)\). Then (3) may be written
\[
f'_1 : \frac{1-v^2}{2} = f(u) : 1 \\
f'_2 : i \left( \frac{1-v^2}{2} \right) f(u) : 1 \\
f'_3 : v = f(u) : 1
\]

From the equations we get
\[
f'_1 = \frac{1-v^2}{2} f(u), \\
f'_2 = i \left( \frac{1-v^2}{2} \right) f(u), \tag{4} \\
f'_3 = v f(u).
\]

Integrating (4) with respect to \( u \) and disregarding the constant of integration, for they could be removed by a translation of the curve in space, we get for the equations of the curve
\[
x = \int \frac{1-v^2}{2} f(u) \, du, \\
y = i \int \frac{1-v^2}{2} f(u) \, du, \tag{5} \\
z = \int v f(u) \, du.
\]

Suppose \( v \) is a constant, say \( a \). Then (5) becomes
\[
x = \frac{1-a^2}{2} \int f(u) \, du, \\
y = i \frac{1-a^2}{2} \int f(u) \, du, \\
z = a \int f(u) \, du.
\]

Let us change the parameter of the curve by replacing \( f(u) \, du \) by a new parameter \( \bar{u} \). We have as the equations of the curve
\[
x = \frac{1-a^2}{2} \bar{u}, \\
y = i \frac{1-a^2}{2} \bar{u}, \tag{6} \\
z = a \bar{u}.
\]

For every value of \( a \), (6) is the equations of an imaginary straight line through the origin.
Let us find the envelope of the family of straight lines by eliminating a. We get an imaginary cone with vertex at the origin whose equation is

\[ x^2 \neq y^2 \neq z^2 = 0 \]  

(7)

Every point on the cone is at a zero distance from the vertex and the distance between any two points on a line is zero. The generators of the cone are minimal straight lines. Their direction cosines are proportional to \( \frac{1-a^2}{2} \), \( \frac{1}{\sqrt{a^2}} \), and a, where a is arbitrary. Therefore through any point in space there are an infinity of minimal straight lines, the locus of which is the cone whose vertex is at the point and whose generators pass through the circle at infinity. To prove the last statement let us consider the equation of a sphere of unit radius and center at the origin in homogeneous coordinates

\[ x^2 \neq y^2 \neq z^2 = w^2. \]

We see that the equations of the circle at infinity are

\[ x^2 \neq y^2 \neq z^2 = 0, \quad w = 0. \]

Therefore (7) passes through the circle at infinity.

Let us now consider the case where the common ratio \( v \) in (2) is a function of \( u \). Let us take this function of \( u \) as the new parameter of the curve and for convenience call it \( \eta \). Then the common ratio in (3) is a function of \( u \), say \( F(\eta) \). Consequently equations (5) may be written

\[ x = \int \frac{1-u^2}{2} F(u) \, du, \]
\[ y = i \int \frac{1-u^2}{2} F(u) \, du, \]
\[ z = \int u F(u) \, du, \]

(8)

where it is obvious from (3) that \( F(u) \) cannot be zero. Let us replace \( F(u) \) by the third derivative of a function \( f(u) \), i.e.,

\[ F(u) = f'''(u). \]
Then equations (8) when integrated by parts become

\[ x = \frac{1}{2}(1-u^2) f''(u) + u f'(u) - f(u), \]
\[ y = \frac{1}{2}(1 + u^2) f''(u) - iu f'(u) - i f(u), \]  
\[ z = u f''(u) + f'(u). \]

Since \( F(u) \) must be different from zero, \( f(u) \) can have any other form than \( c_1 u^2 + c_2 u + c_3 \), where \( c_1, c_2, c_3 \) are arbitrary constants.

2. Minimal Curves on a Surface. Let the equation of a surface be

\[ x = x(u, v), \quad y = y(u, v), \quad z = z(u, v). \]  

An ordinary differential equation of the type

\[ R \frac{du^2}{2} + S \frac{dudv}{2} + T dv^2 = 0, \]  

where \( R, S \) and \( T \) are functions of \( u \) and \( v \), is equivalent to two ordinary differential equations of the first degree, obtained by solving (11) as a quadratic in \( dv \). Therefore equation (11) represents two families of curves upon the surface (10). If the first fundamental form of the surface (10) is set equal to zero, i.e.,

\[ E \frac{du^2}{2} + F \frac{dudv}{2} + G dv^2 = 0, \]  

the resulting equation defines the double family of imaginary curves of length zero, which lie upon the surface (10). For the linear element on the surface is given by

\[ ds^2 = E \frac{du^2}{2} + 2F \frac{dudv}{2} + G dv^2. \]

Theorem 1. Minimal lines on a surface are orthogonal, if the surface is an isotropic developable.

Proof: Let us seek the condition that the minimal lines given by (12) should form an orthogonal system. Let \( K_1 \) and \( K_2 \) be two values of \( \frac{dv}{du} \) obtained from (12). Then by Theory of Equations

\[ K_1 \neq K_2 = \frac{2F}{G}, \]  
\[ K_1 K_2 = \frac{E}{G}. \]
Let \( \theta \) denote the angle between the positive direction of the two minimal curves at their point of intersection. Then if \( \alpha, \beta, \gamma \) and \( \alpha', \beta', \gamma' \) are the direction-cosines of the tangents at each of the two curves at their common point,

\[
\cos \theta = \frac{d\alpha}{d\beta} \frac{\beta'}{\gamma'} = \frac{Edu}{Fs} F(du \neq dv) F Gdv \frac{dv}{ds}, \tag{14}
\]

where \( \delta \) indicates variations in the direction of one of the curves and \( d \) indicates the variation in the direction of the other curve. If the curves are orthogonal \( \theta \) is 90° and consequently \( \cos \theta \) is zero. Hence by (14)

\[
Edu F(du \neq dv) Gdv = 0.
\]

Therefore we have,

Theorem 2. A necessary and sufficient condition that the tangent to two curves upon a surface at a point of meeting be perpendicular is

\[
Edu F(du \neq dv) Gdv = 0
\]

or,

\[
E F \left( \frac{dv}{du} + \frac{dv}{du'} \right) G \frac{dv}{du} \frac{dv}{du'} = 0. \tag{15}
\]

Let \( \frac{dv}{du} = K_1 \), the direction along one curve at the common point, and \( \frac{dv}{du} = K_2 \), the direction along the other curve at the common point.

Then by (13), (15) becomes

\[
E F (K_1 + K_2) G K_1 K_2 = 0
\]

or,

\[
E F \left( \frac{-2F}{G} \right) G \frac{E}{G} = 0
\]

or,

\[
EG - F^2 = 0. \tag{16}
\]

But if

\[
\sqrt{EG - F^2} = 0
\]

the surface is an isotropic developable and vice versa. Therefore the
minimal curves on a surface form an orthogonal system only when the surface is an isotropic developable. Let us illustrate by finding the minimal curves on a unit sphere with center at the origin. The equation of the sphere is

$$x^2 + y^2 + z^2 = 1,$$

which may be written in either of the forms

$$\frac{x + iy}{1 - z} = \frac{1}{1 - iy},$$

or

$$\frac{x - iy}{1 - z} = \frac{1}{x - iy},$$

where \(u\) and \(v\) denote the respective ratios and are conjugate imaginaries.

Solving (17) for \(x, y\) and \(z\) we get

$$x = \frac{u}{uv - 1}, \quad y = \frac{v}{uv - 1}, \quad z = \frac{uv - 1}{uv - 1}$$

(18) is the parametric equation of the unit sphere.

From (18) we can compute the linear element of the surface, i.e.,

$$ds^2 = E\, du^2 + 2F\, du\, dv + G\, dv^2.$$  (19)

For

$$E = \sum x_u^2, \quad F = \sum x_u x_v, \quad G = \sum x_v^2, \quad x_u = \frac{1 - v^2}{(uv - 1)^2}, \quad x_v = \frac{1 - u^2}{(uv - 1)^2},$$

$$y_u = \frac{-i(1/u^2)}{(uv - 1)^2}, \quad y_v = \frac{i(1/v^2)}{(uv - 1)^2}, \quad z_u = \frac{2v}{(uv - 1)^2}, \quad z_v = \frac{2u}{(uv - 1)^2}.$$

Therefore

$$E = \sum x_u^2 = \left(\frac{1 - v^2}{(uv - 1)^2}\right)^2 - \left(\frac{i(1/v^2)}{(uv - 1)^2}\right)^2 \left(\frac{2v}{(uv - 1)^2}\right)^2 = 0,$$

also

$$F = \sum x_u x_v \left(\frac{1 - v^2}{(uv - 1)^2}\right) \left(\frac{1 - u^2}{(uv - 1)^2}\right) + \left(\frac{2v}{(uv - 1)^2}\right) \left(\frac{2u}{(uv - 1)^2}\right),$$

and

$$G = \sum x_v^2 = \left(\frac{1 - u^2}{(uv - 1)^2}\right)^2 - \left(\frac{i(1/v^2)}{(uv - 1)^2}\right)^2 \left(\frac{2u}{(uv - 1)^2}\right)^2 = 0.$$
Substituting the above values of $E, F,$ and $G$ in (19) we get

$$ds^2 = \frac{4dudv}{(uv^2/1)^2}$$

(20)

If $ds^2 = 0$,

we have

$$dudv = 0$$

$$du = 0$$

and

$$dv = 0$$

From the last two equations it is obvious that the parametric curves $u = \text{constant}$ and $v = \text{constant}$ are the minimal curves on the sphere, i.e., they are lines of zero length.

In order to obtain the equation of the minimal lines let us eliminate $u$ from the first two and the last two of equation (18).

Solving the first equation of (18) for $u$, we get

$$u = \frac{v-x}{vx-1}$$

Substituting this value of $u$ in the second equation of (18) we get after simplifying

$$i(v^2 - 1) x y (1 - v^2)y - 2iv = 0.$$  (21)

Solving the last equation of (18) for $u$, we get

$$u = \frac{1/z}{v-vz}$$

Substituting this value of $u$ in the second equation of (18) we obtain after reduction

$$i(v^2 - 1) z f 2vy f i (1 - v^2) = 0.$$  (22)

Equations (21) and (22) show that all the points of the curve $v = \text{constant}$ lie on the line

$$i(v^2 - 1) x y (1 - v^2)y - 2iv = 0$$

$$i(v^2 - 1) Z f 2vy f i (1 - v^2) = 0.$$  (23)
where \( X, Y, Z \) denote current coordinates. By the use of (18) these equations can be written

\[
\frac{X - x_o}{v - 1} = \frac{Y - y_o}{i(\nu / \lambda)} = \frac{Z - z_o}{-2v}.
\]

where \( x_o, y_o, z_o \) are the coordinates of a particular point. Also let us eliminate \( v \) from the first two and the last two equations of (18). Solving the first equation for \( v \) we get

\[
v = \frac{u-x}{xu-1}.
\]

Substituting this value in the second equation and simplifying we get

\[
i(1 / u^2)x^2 (u^2 - 1)y - 2iu = 0. \quad (24)
\]

Solving the last equation for \( v \) we obtain

\[
v = \frac{1 / z}{u(1 - z)}.
\]

Substituting this value in the second equation and simplifying we obtain

\[
i(1 / u) Z - 2uv \neq 1 (1 - u) = 0. \quad (25)
\]

Equations (24) and (25) show that all the points of the curve \( u \) constant lie on the line.

\[
i(1 / u^2)x \neq (u^2 - 1) Y - 2iu = 0 \quad (26)
\]

\[
i(1 / u^2)Z - 2uv \neq 1 (1 - u^2) = 0
\]

where \( X, Y, Z \) denote current coordinates. By the use of (18) these equations can be written

\[
\frac{X - x_o}{1 - u^2} = \frac{Y - y_o}{i(1 / u^2)} = \frac{Z - z_o}{2u},
\]

where \( x_o, y_o, z_o \) are coordinates of a particular point.
CHAPTER IV
MINIMAL SURFACES

1. The General Case. Let us consider the problem of Lagrange:

Given a closed curve C and a connected surface S bounded by the curve:

To determine S so that the enclosed area may be a minimum.

If the surface be defined by the equation \( z = f(x, y) \), the problem requires the determination of \( f(x, y) \) so that the integral \( \iint \sqrt{1 + p^2 + q^2} \, dx \, dy \)

extended over the portion of the surface bounded by C shall be a minimum, where \( p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y} \).

Lagrange showed that the condition for this integral to be a minimum is

\[
\frac{\partial}{\partial x} \left( \frac{p}{1 + p^2 + q^2} \right) - \frac{\partial}{\partial y} \left( \frac{q}{1 + p^2 + q^2} \right) = 0 \quad (1)
\]

that is

\[
\sqrt{1 + p^2 + q^2} = p(1 + p^2 + q^2)^{-\frac{1}{2}} \left( \frac{\partial^2 z}{\partial x^2} + q \frac{\partial^2 z}{\partial y^2} \right) - q(1 + p^2 + q^2)^{-\frac{1}{2}} \left( \frac{\partial^2 z}{\partial x^2} + q \frac{\partial^2 z}{\partial y^2} \right) = 0
\]

or,

\[
(1 + q^2)r - 2pq s (1 + p^2) t = 0, \quad (2)
\]

where \( r, s, t \) are equal to \( \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2} \) and \( \frac{\partial z}{\partial y} \), respectively.

Later Meusnier proved that equation (2) is equivalent to the vanishing of the mean curvature

\[
K_m = \frac{1}{r} + \frac{1}{s} \left( \frac{\partial^2 z}{\partial x^2} + q \frac{\partial^2 z}{\partial y^2} \right) = \frac{ED'' - CD^2 - 2FD'}{H^2}.
\]

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Therefore the surfaces furnishing a solution of Lagrange's problem are characterized by the geometric property that their mean curvatures are zero. We shall use this property as a definition of minimal surfaces. At each point of a minimal surface then, the principal radii differ only in sign and every point is a hyperbolic point and its Dupin Indicatrix is a equilateral hyperbola. Therefore minimal surfaces are characterized by the property that their asymptotic lines form an orthogonal system. Also, the tangents to two asymptotic lines at a point bisect the angles between the lines of curvature at the point and vice versa.

From Chapter II section 3 we saw that the fundamental quantities of a surface had the following relations:

\[ \begin{align*}
\mathcal{E} &= K_m D - KE, \\
\mathcal{F} &= K_m D' - KE, \\
\mathcal{G} &= K_m D'' - KG.
\end{align*} \] (3)

From (3) we see that the linear element for the spherical representation of the surface is given by

\[ ds^2 = \mathcal{E} du^2 + 2\mathcal{F} du dv + \mathcal{G} dv^2 \left( \frac{K}{R} - K \right) ds^2. \] (4)

If the surface is a sphere \( R \) is a function of \( u \) and \( v \). When the surface is minimal,

\[ ds^2 \geq 0. \]

Therefore from (4)

\[ ds^2 = \varphi(u,v)ds^2, \]

when the surface is a sphere or minimal. Hence we have

**Theorem 1.** A necessary and sufficient condition that the spherical representation of a surface be conformal is that it be minimal of a sphere.

Therefore isothermal orthogonal systems on the surface are represented by similar systems on the sphere, and conversely.
From the sections on minimal curves on a surface we know that all isothermal orthogonal systems on the sphere are known. Suppose that one of these systems are parametric and that the linear element is
\[ d\sigma^2 = \lambda(du^2 \neq dv^2). \tag{5} \]

But from Chapter II section 3 we saw that the mean curvature
\[ \rho_1 \rho_2 = \frac{\rho \rho' + \rho' \rho - 2 \rho \rho'}{\rho^2}. \]

Hence this vanishes, i.e.,
\[ \rho \rho' - \rho \rho' - 2 \rho \rho' = 0 \tag{6} \]

this is the condition for the spherical representation to be a minimal surface. Hence if (5) is the square of the linear element of the spherical representation, we see that
\[ \rho = \rho' = \lambda \]

and
\[ \rho = 0. \]

Hence in this case from (6) we have
\[ D \neq D'' = 0 \tag{7} \]

By means of (7) the Codazzi equation of condition for a surface, viz.,
\[ \frac{\partial D}{\partial v} - \frac{\partial D'}{\partial u} = 0 \quad \frac{\partial D}{\partial w} - \frac{\partial D'}{\partial \omega} = 0 \]
\[ \frac{\partial D'}{\partial v} - \frac{\partial D}{\partial u} = 0 \quad \frac{\partial D'}{\partial w} - \frac{\partial D}{\partial \omega} = 0 \tag{8} \]

are reducible to
\[ \frac{\partial D}{\partial v} - \frac{\partial D'}{\partial u} = 0 \tag{9} \]

By eliminating D or D' we find that both D and D' are integrals of the
equation
\[
\frac{\partial^2 D}{\partial u^2} + \frac{\partial^2 D}{\partial v^2} = 0 \quad (10)
\]
this shows that \( D \) and \( D' \) are harmonic functions. \( (10) \) can be verified in the following manner: Differentiate the first equation in \( (9) \) with respect to \( u \) and get
\[
\frac{\partial^2 D}{\partial u^2} - \frac{\partial^2 D'}{\partial u^2} = 0
\]
Differentiate the second equation in \( (9) \) with respect to \( v \) and get
\[
\frac{\partial^2 D}{\partial u \partial v} - \frac{\partial^2 D'}{\partial v \partial v} = 0
\]
eliminate \( \frac{\partial^2 D}{\partial u^2} \) and get
\[
\frac{\partial^2 D}{\partial u^2} - \frac{\partial^2 D'}{\partial v^2} = 0.
\]
Hence the most general form of \( D' \) is
\[
D' = \phi(u, iv) \neq \psi(u, iv), \quad (11)
\]
where \( \phi \) and \( \psi \) are arbitrary functions. Now from \( (9) \) we have by integration
\[
D = -D'' = i(\phi - \psi) \neq c, \quad (11)'
\]
where \( c \) is the constant of integration. To each pair of functions \( \phi \) and \( \psi \) there corresponds a minimal surface, whose cartesian coordinates are given by
\[
\frac{\partial x}{\partial u} = -\frac{1}{i} \left( D' \frac{\partial x}{\partial u} \neq D'' \frac{\partial x}{\partial u} \right), \quad (12)
\]
and similar expressions in \( y \) and \( z \). Equation \( (12) \) is easily deduced from the general case, viz.
\[
\frac{\partial x}{\partial u} = \frac{TD' - yD}{N^2} \frac{\partial x}{\partial u} + \frac{TD' - e D'}{N^2} \frac{\partial x}{\partial v}
\]
and
\[
\frac{\partial x}{\partial v} = \frac{TD'' - yD'}{N^2} \frac{\partial x}{\partial u} + \frac{TD' - e D'}{N^2} \frac{\partial x}{\partial v}.
\]
Evidently the surface is real only when \( \varphi \) and \( \psi \) are conjugate functions.

In the preceding results we have assumed that neither \( D \) nor \( D' \) is zero. We notice however, that either may be zero and then the other is a constant, which is zero only for the plane. Therefore we have

Theorem 2. Every isothermal system on the sphere is the representation of the lines of curvature of a unique minimal surface and of the asymptotic lines of another minimal surface.

The converse is also true: The spherical representations of the lines of curvature and of the asymptotic lines of a minimal surface are isothermal systems.

Proof: If the lines of curvature are parametric (6) may be replaced by \( D = \rho \xi \) and \( D'' = -\rho \gamma \), where \( \rho \) is equal to either principal radius to within its algebraic sign. When these values and \( D' = \varphi = 0 \) are substituted into the Codazzi equations (9) we obtain

\[
\frac{\partial}{\partial y}(\rho \xi) = 0
\]

and

\[
\frac{\partial}{\partial \mu}(\rho \gamma) = 0,
\]

so that,

\[
\frac{\xi}{\gamma} = \frac{\mu}{\nu}.
\]

When the asymptotic lines are parametric we have

\( D = D'' = \varphi = 0 \)

and the Codazzi equations reduce to

\[
\frac{\partial}{\partial \mu} \left( \sqrt{\frac{\xi}{\gamma}} D \right) = 0, \quad \frac{\partial}{\partial \nu} \left( \sqrt{\frac{\xi}{\gamma}} D' \right) = 0.
\]

From which it follows that \( \frac{\xi}{\gamma} = \frac{\mu}{\nu} \).

2. Lines of Curvature and Asymptotic Lines. Adjoint minimal surfaces.
Let us consider equation (9) section 1 and investigate the minimal surface with its lines of curvature represented by an isothermal system. We may take without loss of generality

$$D = -D'' = 1, \quad D' = 0.$$  \hspace{1cm} (0.3)

From the total curvature for the spherical representation we have in this case

$$\frac{1}{\rho_1 \rho_2} = -\frac{1}{\rho^2} = \lambda^2, \quad E = G = \rho,$$

where \(\rho = |\rho_1| = |\rho_2|.

Hence we have

Theorem 3. The parameters of the lines of curvature may be so chosen that the linear elements of the surface and of its spherical representation have the respective forms

$$ds^2 = \rho (du^2 + dv^2),$$

$$d\sigma^2 = \frac{1}{\rho} (du^2 + dv^2),$$

where \(\rho\) is the absolute value of each principal radius.

Also we may take for the solution of (9) section 1.

$$D = D'' = 0, \quad D' = 1$$  \hspace{1cm} (1.4)

again we find

$$\frac{1}{\rho_1 \rho_2} = -\frac{1}{\rho^2} = -\lambda^2, \quad E = G = \rho$$

so that we have a result similar to the one above: The parameters of the asymptotic lines of a minimal surface may be so chosen that the linear elements of the surface and of its spherical representation have the respected forms

$$ds^2 = \rho (du^2 + dv^2),$$

$$d\sigma^2 = \frac{1}{\rho} (du^2 + dv^2),$$

where \(\rho\) is the absolute value of each principal radius.
From the symmetric form of equation (9) it follows that if equations (11) and (11') section 1 represent one set of solutions, another set is given by
\[ \begin{align*}
D_1 &= -D''_1 = \rho + \psi \\
D'_1 &= i(\rho - \psi) - c.
\end{align*} \]
These values are such that
\[ \begin{align*}
ED''_1 + D' D_1 - 2D' D'_1 &= 0,
\end{align*} \]
which is the condition that asymptotic lines of either surface correspond to a conjugate system of the other. For a necessary and sufficient condition that the curves defined by
\[ \begin{align*}
Rd^2 &= 2S dudv \not\perp T dv^2 = 0
\end{align*} \]
form a conjugate system upon a surface is
\[ \begin{align*}
RD'' &= TD - 2SB' = 0.
\end{align*} \]
When this condition is satisfied by two minimal surfaces and the tangent plane that corresponds to the points are parallel, these two surfaces are said to be adjoints of one another. Hence a pair of functions \( \rho \) and \( \psi \) determines a pair of adjoint minimal surfaces. When in particular the asymptotic lines on one surface are parametric, the second fundamental coefficients have the values (14) and on the other the values (13).

It follows then from (12) section 1 that between the cartesian coordinates of minimal surfaces and their adjoints, the following relations hold:
\[ \frac{\partial x}{\partial u} = \frac{\partial x}{\partial u}, \quad \frac{\partial x}{\partial v} = -\frac{\partial x}{\partial u} \]
and similar expressions in the \( y \)'s and \( z \)'s, when the parametric curves are asymptotic on the locus \( x y z \).

3. Minimal Curves on a Minimal Surface. Let us consider the lines of length zero upon a minimal surface, as the parametric curves. Since the lines of length zero are minimal, or parametric, we have
$$E = G = 0 \quad (15)$$

From (3) section 1, it follows that the parametric lines on the sphere are also minimal lines, i.e., rectilinear generators. From (6) section (1) we find $D' = 0$. Conversely when the latter is zero, and the parametric lines are minimal curves; it follows from the expression for the mean curvature of a surface, viz.,

$$K = \frac{ED' - GD' - 2FD'}{H^2},$$

that $K = 0$. Hence we have

Theorem 4. A necessary and sufficient condition for a surface to be minimal is that the lines of zero length form a conjugate system.

By means of (13) and the analytical condition that the parametric lines form a conjugate system, viz., that $x, y, z$, are solutions of the equation of the type

$$\frac{\partial^2 \phi}{\partial u \partial v} \neq a \frac{\partial \phi}{\partial u} \neq b \frac{\partial \phi}{\partial v} = 0,$$

where $a$ and $b$ are functions of $u$ and $v$ or constants, the point equation of a minimal surface referred to its minimal lines is

$$\frac{\partial^2 \phi}{\partial u \partial v} = 0.$$

Hence the finite equations of the surface are of the forms

$$x = U_1 \neq V_1, \quad y = U_2 \neq V_1, \quad Z = U_3 \neq V_3, \quad (16)$$

where $U_1$, $U_2$, $U_3$ are functions of $u$ alone and $V_1$, $V_2$, $V_3$ are functions of $v$ alone, satisfying the conditions

$$\frac{U_1^2}{V_1^2} \neq \frac{U_2^2}{V_2^2} \neq \frac{U_3^2}{V_3^2} = 0. \quad (17)$$

From (16) it is seen that minimal surfaces are surfaces of translation, and from (17) that the generators are minimal curves. For surfaces of translation are generated by one curve which when translated every point on it describes another curve. And these two curves are the para-
metric curves of the surface.

Lie has observed that the surface defined by (16) is the locus of the midpoints of the joins of points on the curves

\[ x_1 = 2U, \quad y_1 = 2U, \quad z_1 = 2U; \]
\[ x_2 = 2V, \quad y_2 = 2V, \quad z_2 = 2V. \]

It may be that these two sets of equations define the same curve in terms of different parameters. In this case the surface is the locus of the midpoint to all chords of the curve. The general case may be stated in

Theorem 5. The locus of the point which divides in constant ratio, the joins of points on two curves or all the chords of one curve is a surface of translation: In the latter the curve is an asymptotic line of the surface. As a result of this theorem we may state that:

Theorem 6. A minimal surface is the locus of the midpoints of the joins of points on two minimal curves.

In section 1 we found that the cartesian coordinates of any minimal curve are expressible in the form

\[ x = \int (1 - u^2)F(u)du, \]
\[ y = \int (1 - u^2)F(u)du, \]
\[ z = \int uF(u)du. \]  

Therefore by the above theorem the following equations define a minimal surface referred to its minimal lines.

\[ x = \frac{1}{2} \int (1 - u^2)F(u)du - \frac{1}{2} \int (1 - v^2)G(v)dv, \]
\[ y = \frac{1}{2} \int (1 - u^2)F(u)du - \frac{1}{2} \int (1 - v^2)G(v)dv, \]
\[ z = \int uF(u)dv - vG(v)dv, \]

where \( F \) and \( G \) are any analytic functions whatever. Also any minimal surface can be defined by equations of this form. For the only apparent lack of generality it is due to the fact that the algebraic signs of (19) are not determined by equations (17) and consequently the signs preceding terms
in the right hand member of equations (19) could be positive or negative. But it can be shown by suitable change of parameters and of the functions $F$ and $\varphi$, all these cases reduce to (19). Thus for example we consider the surface defined by the equations which result when the terms of the right hand member of (18) are replaced by

$$\frac{1}{2} \int (1 - v^2) \varphi_u (v) dv + \frac{1}{2} \int (1 - v^2) \varphi_v (v) dv + \int v \varphi (v) dv,$$  

(20)

In order that the surface just defined can be brought into coincidence, by a translation with the surface (19) we must have

$$(1 - v^2) \varphi_u dv = (1 - v^2) \varphi dv,$$

$$(1 - v^2) \varphi_v dv = -(1 - v^2) \varphi dv,$$

$$v \varphi dv = v \varphi dv.$$

Dividing these equations, member by member we have

$$\frac{1 - v^2}{1 - v^2} = -\frac{1 - v^2}{1 - v^2} = \frac{v}{v},$$

from which it follows that

$$v = -\frac{1}{v}.$$  

Substituting this value in the last of the above equations, we find

$$\varphi (v) = -v^2 \varphi (v),$$

and this value satisfies the other equations. Similar results follow when another choice of signs is made. The reason for the particular choice made in (19) will be seen when we discuss the reality of the surfaces. Therefore we have proved

Theorem 7. When a minimal surface is defined by equation (19), the necessary and sufficient condition that the two generating curves be superposable by a translation, is that

$$F(u) = \frac{1}{u^2} \varphi (u).$$

(21)

From (19) we obtain

$$E = 0, \quad F = \frac{1}{2} \int (1 - uv)^2 F(u) \varphi (v), \quad G = 0,$$

so that the linear element is
\[ ds^2 = (1 - uv)^2 F(u) \Phi(v) \, dudv. \]  

We find for the expressions of the direction cosines of the normal

\[ X = \frac{u}{1 - uv}, \quad Y = \frac{i(v - u)}{1 - uv}, \quad Z = \frac{uv - 1}{1 - uv}, \]  

and the linear element of the spherical representation is

\[ ds^2 = \frac{4 \, dudv}{(1 - uv)^2}. \]

Also we have

\[ D'' = -\Phi(u), \quad D' = 0, \quad D = -F(u), \]  

so that the equations of the lines of curvature and of the asymptotic lines are respectively

\[ F(u) \, du^2 - \Phi(v) \, dv^2 = 0, \]  

\[ F(u) \, du^2 + \Phi(v) \, dv^2 = 0. \]

These equations are of such a form that we have

**Theorem 8.** When the minimal surface is referred to its minimal lines, the finite equations of the lines of curvature and asymptotic lines are given by quadratures which are the same in both cases.

In order that a surface be real its spherical representation must be real. Consequently \( u \) and \( v \) must be conjugate imaginaries as is seen from (23) and the functions \( F \) and \( \Phi \) must be conjugate imaginaries. Hence if \( R \theta \) denotes the real part of the function \( \theta \), all real minimal surfaces are defined by

\[ x = R \int (1 - u^2) \, du, \]

\[ y = R \int i(1 - u^2) F(u) \, du, \]

\[ z = R \int 2u F(u) \, du, \]

where \( F \) is any function whatever of a complex variable \( u \). In like manner the equations of the line of curvature may be written in the form

\[ R \int \sqrt{F(u)} \, du = \text{Constant}, \quad R \int i \sqrt{F(u)} \, du = \text{Constant}, \]  

(27)
4. Double Minimal Surfaces. Let us inquire whether the same minimal surface can be defined in more than one way by equations of the form (19) section 3. We are assuming that this is possible and indicate by \( u_i, v_i \), and \( F_i(u), \Phi_i(v) \) the corresponding parameters and functions. As the parameters \( u_i \) and \( v_i \) refer to the lines of length zero on the surface, each is a function of either \( u \) or \( v \). In order to determine the forms of the latter we make use of the fact that the positive directions of the normal to the surface in the two forms of parametric representation may have the same or opposite senses. When they have the same sense equations (23) and similar ones in \( u_i \) and \( v_i \) must be equal respectively. In this case

\[
u_i = u_i, \quad v_i = v\] (28)

If the senses are opposite the respective expression are equal to within algebraic signs. From the resulting equations, we find

\[
u_i = -\frac{1}{v}, \quad v_i = -\frac{1}{u}\] (29)

If we compare equations (19) section 3 with analogous equations in \( u, v \) we find that for the case (28) we must have

\[
F_i(u_i) = F(u), \quad \Phi_i(v_i) = \Phi(v)
\]

and for the case (29)

\[
F_i(u_i) = -v^* \Phi(v_i), \quad \Phi_i(v_i) = -u^* F(u_i).
\]

Hence we have

Theorem 9. A necessary and sufficient condition that two minimal surfaces determined by the pairs of functions \( F, \Phi \) and \( F_i, \Phi_i \), be congruent is that

\[
F_i(u) = -\frac{v}{u^*} \Phi(-\frac{u}{v}), \quad \Phi_i(v) = -\frac{v}{v^*} F_i(-\frac{v}{u}); \quad (30)
\]

to the point \((u, v)\) on one surface corresponds the point \((-\frac{1}{v}, -\frac{u}{v^*})\) on the other and the normals at these points are parallel but of different
sense.

In general the functions $F$ and $F_i$ as given by (30) are not the same. If they are the same, then $\Phi$ and $\Phi_i$ are the same. In this case the right hand members of equations (19) section 3 are unaltered, when $u$ and $v$ are replaced by $-\frac{1}{v}$ and $\frac{1}{u}$, respectively. Hence the cartesian coordinates of the points $(u,v)$ and $(-\frac{1}{v}, -\frac{1}{u})$ differ at most by constants. Therefore the regions of the surface about these points either coincide or can be brought into coincidence by a translation. In the latter case, the surface is periodic and consequently transcendental.

Suppose that it is not periodic and consider a point $P_o(u_o, v_o)$. As $u$ varies continuously from $u_o$ to $-\frac{1}{u_o}$, $v$ varies from $v_o$ to $-\frac{1}{v_o}$ and the point $P_o$ describes a closed curve by returning to $P_o$. But now the positive normal is on the other side of the surface. Hence these surfaces have the property that a point can pass continuously from one side to the other without going thru the surface. On this account they are called double minimal surfaces.

From theorem 7 section 3, it follows, that double minimal surfaces are characterized by the property that the minimal curves in both system are superposable by a translation. The equations of such a surface may be written.

$$
\begin{align*}
x &= \frac{1}{f_1(u)} \neq f_1(v), \quad y = \frac{3}{f_2(u)} \neq f_2(v), \quad z = \frac{3}{f_3(u)} \neq f_3(v).
\end{align*}
$$

The surface is therefore the locus of the midpoints of the chords of the curve

$$
\begin{align*}
\xi &= f_1(u), \quad \eta = f_2(u), \quad \xi = f_3(u),
\end{align*}
$$

which lies upon the surface and is the envelope of the parametric curves.

5. Algebraic Minimal Surfaces. Let us investigate as did Weirstrass, the possibility of putting the formulas of (19) section 3 in a form free
of all quadratures. This can be done by putting $f''(u)$ and $\varphi''(v)$ in place of $F(u)$ and $\Phi(v)$, where the accents indicate differentiation, and then integrating by parts. We get therefore

$$
\begin{align*}
x &= \frac{1}{2} - \frac{u^2}{2} f''(u) - u f'(u) - \frac{1 - \nu^2}{2} \varphi''(v) - \nu \varphi'(v) - \varphi(v) - f'(\omega), \\
y &= \frac{1}{2} (1 - u^2)f''(u) - iuf'(u) - if(u) - \frac{1}{2} (1 - \nu^2) \varphi''(v) - iv \varphi'(v) - i \varphi(v), \\
z &= uf''(u) - \nu \varphi(v) - \varphi'(v) - f'(\omega).
\end{align*}
$$

(31)

It is clear that the surface so defined is real when $f$ and $\varphi$ are conjugate imaginary functions. In this case the equations (31) may be written

$$
\begin{align*}
x &= R \left[ (1 - u^2) f''(u) - 2u f'(u) - 2f(u) \right], \\
y &= R i \left[ (1 - u^2) f''(u) - 2uf'(u) - 2f(u) \right], \\
z &= R \left[ 2uf''(u) - 2f'(u) \right].
\end{align*}
$$

(32)

However it is not necessary that $f$ and $\varphi$ be conjugate imaginary or that the surface be real. For equations (31) are unaltered if $f$ and $\varphi$ are replaced by

$$
\begin{align*}
f_i(u) &= f(u) - A(1 - u^2) - B \nu (1 - u^2) - 2Cu, \\
\varphi_i(v) &= \varphi(v) - A(1 - \nu^2) - B \nu (1 - \nu^2) - 2C\nu,
\end{align*}
$$

where $A$, $B$, $C$ are any constants whatever. Evidently if $f$ and $\varphi$ are conjugate imaginaries the same is not in general true of $f_i$ and $\varphi_i$; but the surface was real for $f$ and $\varphi$ and consequently it will be real for $f_i$ and $\varphi_i$.

Formulas (31) are of particular value in the study of algebraic surfaces. Therefore the surfaces are algebraic when $f$ and $\varphi$ are algebraic. It can be shown that every algebraic minimal surface is determined by algebraic functions of $f$ and $\varphi$.

6. Associate Minimal Surfaces. When the equations of a minimal surface $S$ are written in the abbreviated form given by equation (16) section 3,
the linear element is
\[ ds^2 = 2\left( dU, dV, dU_2 dV_2, dU_3 dV_3 \right). \]
This is the linear element also of a surface defined by
\[ x_\alpha = e^{i\phi} u_\alpha + e^{-i\phi} v_\alpha, \quad y_\alpha = e^{i\phi} u_\alpha + e^{-i\phi} v_\alpha, \quad z_\alpha = e^{i\phi} u_\alpha + e^{-i\phi} v_\alpha \tag{33} \]
where \( \phi \) is any constant. There are an infinity of such surfaces called associate minimal surfaces. It is readily found that the direction cosines of the normal to any one have the values found in section 1. Hence any two associate minimal surfaces defined by (33) have their tangent planes at corresponding points parallel and are applicable.

Of particular interest is the surface \( S \), for which \( \phi = \nu \).

Its equations are
\[ x = \frac{1}{2} \int (1 - u^2) F(u) du - \frac{i}{2} \int (1 - v^2) \Phi(v) dv, \]
\[ y = -\frac{1}{2} \int (1 - u^2) F(u) du - \frac{i}{2} \int (1 - v^2) \Phi(v) dv, \tag{34} \]
\[ z = i \int u F(u) du - i \int v \Phi(v) dv. \]

In order to show that \( S \) is the adjoint minimal surface of \( S \) (see section on adjoint minimal surfaces) we have only to prove that the asymptotic lines on one correspond to the lines of curvature on the other. For \( S \), the equations of the lines of curvature and asymptotic lines are
\[ i F(u) du^2 - i \Phi(u) du^2 = 0 \]
and
\[ i F(u) du^2 - i \Phi(v) dv^2 = 0, \]
respectively. Comparing these with the equations for the lines of curvature and the asymptotic lines for the minimal surface \( S \) as given in section 1, we see that the desired condition is satisfied.

From equations (19) section 3 and (34) we obtain identities:
\[ dx^2 \neq dy^2 \neq dz^2 \neq dx^2 \neq dy^2 \neq dz^2 \tag{35} \]
\[ dx dx_i \neq dy dy_i \neq dz dz_i = 0. \]
The later equation has the following interpretation: On two adjoint minimal surfaces at points corresponding to parallelism of tangent planes, the tangents to corresponding curves are perpendicular.

From equation (27) section 3 it follows that if we put
\[ u = \int i\varphi - \sqrt{F} \, du, \]
the curves \( u = \text{constant} \) and \( \varphi = \text{constant} \) are its lines of curvature. Moreover for an associate surface the lines of curvature are given by
\[
R \left[ \dot{\xi} \left( u \right) + i \varphi \right] = \text{constant}, \quad R \left[ \dot{\xi} \left( \varphi \right) + i u \right] = \text{constant},
\]
or,
\[
u \cos \frac{\alpha}{2} \varphi \sin \frac{\alpha}{2} = \text{constant}, \quad \nu \sin \frac{\alpha}{2} \varphi \cos \frac{\alpha}{2} = \text{constant}.
\]
From this result we have

**Theorem 10.** The lines of curvature on a minimal surface \( S \) associate to a surface \( S \) correspond to the curves on \( S \) which cut its lines of curvature under the constant angle \( \frac{\alpha}{2} \).

Since equation (33) may be written
\[
x_\alpha = x \cos \alpha - x \sin \alpha, \\
y_\alpha = y \cos \alpha - y \sin \alpha, \\
z_\alpha = z \cos \alpha - z \sin \alpha
\]
the plane determined by the origin of coordinates, a point \( P \) on a minimal surface and the corresponding point on its adjoint, contains the point \( P^\alpha \) corresponding to \( P \) on every associate minimal surface. Eliminating \( \alpha \) from (36) we find that the locus of these points is an ellipse with center at origin. Let us combine this result with the result obtained in the first part of this section, and get

**Theorem 11.** A minimal surface admits of a continuous deformation
into a series of minimal surfaces, and each point of the surface describes an ellipse whose plane passes through a fixed point which is the center of the ellipse.

7. The Formulas of Schwarz. Since the tangent planes to a minimal surface and its adjoint at corresponding points are parallel, we have

\[ Xdx_i \neq Ydy_i \neq Zdz_i = 0. \]

From this and the second equation in (35) section 6, we obtain the proportion

\[ \frac{dx_i}{Zdy - Ydz} = \frac{dy_i}{Xdz - Zdx} = \frac{dz_i}{Ydx - Xdy}. \] (37)

By means of the first equation of (35) section 6, the sums of the squares of the numerators and of the denominators are equal. Therefore the common ratio is ±1. If the expressions for the various quantities be substituted from (19) section 3, and the equations for the direction cosines for the normal to the minimal surface and equation (34) section 6, we find that the value for the common ratio in (37) is -1. Hence we have,

\[ dx_i = Ydz - Zdy, \quad dy_i = Zdx - Xdz, \quad dz_i = Xdy - Ydx \] (38)

From these equations and (16) section 3, and (34) section 6, we have

\[ 2U_1 = x - ix_i = x - i \int Zdy - Ydz, \] (39)

\[ 2U_2 = y - iy_i = y - i \int Xdz - Zdx, \]

\[ 2U_3 = z - iz_i = z - i \int Ydx - Xdy, \]

and

\[ 2V_1 = x - ix_i = x - i \int Zdy - Ydz, \]

\[ 2V_2 = y - iy_i = y - i \int Xdz - Zdx, \] (40)

\[ 2V_3 = z - iz_i = z - i \int Ydx - Xdy. \]

These equations (39) and (40) are known as the formulas of Schwarz. Their importance is due to their ready applicability to the solution of the problem:
To determine a minimal surface passing through a given curve and admitting at each point of the curve a given tangent plane.

In solving this problem let C be a curve whose coordinates x, y and z are analytic functions of a parameter t, and let X, Y, Z be analytic functions of t satisfying the conditions

\[ X^2 + Y^2 + Z^2 = 1, \quad Xdx + Ydy + Zdz = 0. \]

If \( x_u, y_u, z_u \) denote the values of \( x, y, z \) when \( t \) is replaced by a complex variable \( u \) and \( x_v, y_v, z_v \) the values when \( t \) is replaced by \( v \), the

\[
\begin{align*}
\bar{x} &= U_1 x - U_2 y + i \frac{x_u}{2} \int_v^u (y - Y) \, du, \\
\bar{y} &= U_2 x - U_3 y + i \frac{x_v}{2} \int_v^u (x - X) \, du, \\
\bar{z} &= U_3 x - U_1 y + i \frac{x_v}{2} \int_v^u (Z - Z) \, du,
\end{align*}
\]

(41)

define a minimal surface which passes through C and admits at each point for a tangent plane the plane through the point with direction - cosine \( X, Y, Z \). For when \( u \) and \( v \) are replaced by \( t \) these equations define \( C \), and the conditions given by (17) section 3, and

\[ \sum X \, du = 0, \sum X \, dv = 0 \]

are satisfied. Furthermore the surface defined by (41) affords the unique solution as is seen from (39) and (40).

When, in particular \( C \) and \( t \) are real the equations of the real minimal surface satisfying the conditions of the problem may be put in the form

\[
\begin{align*}
\bar{x} &= R [x - i \int_x^u (Zdy - Ydx)] , \\
\bar{y} &= R [y + i \int_x^u (Xdz - Zdx)] , \\
\bar{z} &= R [z + i \int_x^u (Ydx - Xdy)] .
\end{align*}
\]

As an application of these formulas we consider minimal surfaces containing a straight line. If we take the straight line for the \( z \)-axis
and let $\varphi$ denote the angle which the normal to the surface at a point of the line makes with the $z$-axis, we have

$$x = y = 0, \quad z = t, \quad X = \cos \varphi, \quad Y = \sin \varphi, \quad Z = 0.$$  

Hence the equations of the surface are

$$\bar{x} = -R \int_x^z \sin \varphi \, dt, \quad \bar{y} = R \int_x^z \cos \varphi \, dt, \quad Z = R.$$  

Here $\varphi$ is an analytic function of $t$ whose form determines the character of the surface. For two points correspond to conjugate values of $u$, the $z$ coordinates are equal and the $x$ and $y$ coordinates differ in sign. Hence: Every straight line upon a minimal surface is an axis of symmetry.