ON

COMPLEX INTEGRATION

A THESIS

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>Definitions and Terminology</td>
<td>1</td>
</tr>
<tr>
<td>II. LINE INTEGRALS</td>
<td>4</td>
</tr>
<tr>
<td>Definition of a Line Integral</td>
<td>4</td>
</tr>
<tr>
<td>Plane Area as a Line Integral</td>
<td>7</td>
</tr>
<tr>
<td>Green's Theorem</td>
<td>10</td>
</tr>
<tr>
<td>III. CAUCHY'S FIRST INTEGRAL THEOREM</td>
<td>14</td>
</tr>
<tr>
<td>Integral Taken Along a Closed Curve</td>
<td>14</td>
</tr>
<tr>
<td>Elementary Proof of Cauchy's Theorem</td>
<td>14</td>
</tr>
<tr>
<td>General Form of Cauchy's Theorem</td>
<td>16</td>
</tr>
<tr>
<td>IV. CAUCHY'S SECOND INTEGRAL THEOREM</td>
<td>21</td>
</tr>
<tr>
<td>Development of Cauchy's Second Integral</td>
<td>21</td>
</tr>
<tr>
<td>Theorem</td>
<td>21</td>
</tr>
<tr>
<td>Derivative of an Analytic Function</td>
<td>23</td>
</tr>
<tr>
<td>V. POLES AND RESIDUES</td>
<td>26</td>
</tr>
<tr>
<td>Poles; Singular Points and Zero Points, Order of Poles</td>
<td>26</td>
</tr>
<tr>
<td>Residues; Cauchy's Theorem of Residues, Residues from Laurent's Expansion, The Residue for ( f(Z) = \frac{P(Z)}{Q(Z)} )</td>
<td>28</td>
</tr>
<tr>
<td>VI. EVALUATION OF REAL INTEGRALS</td>
<td>33</td>
</tr>
<tr>
<td>Evaluation of Definite Integrals of the form:</td>
<td></td>
</tr>
<tr>
<td>[ \int_{-\infty}^{\infty} f(x) , dx ], [ \int_{0}^{2\pi} f(\sin x, \cos x) , dx ]</td>
<td>33</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>44</td>
</tr>
</tbody>
</table>
CHAPTER I

INTRODUCTION

Too often the student of the calculus of real variables is confronted with difficulty in integrating certain problems involving a real variable. This difficulty is sometimes overcome when the student has a knowledge of the calculus of complex variables. That is to say, the student does not rely wholly upon the axis of reals as his path of integration but, rather, he moves into the complex plane, which contains the real axis and the imaginary axis.

In chapter II we will discuss line-integral, which is basic to the consideration of complex integration. In chapters III and IV we will discuss Cauchy's First Integral theorem and Cauchy's Integral Formula, with major emphasis on their developments and their proofs. Poles and Residues will be considered in chapter V, and in chapter VI we give the development and the solution of some of the problems involving real integrals by complex integration.

Definitions and Terminology

A function is said to be single valued when one and only one value exists for the independent variable. If a given single valued function $f(Z)$ has a uniquely determined derivative at a point $q$ and at every point in the neighborhood of $q$, then $Z = q$ is called a regular point of $f(Z)$. 

1
A point in every deleted neighborhood of which there are regular points but itself is not a regular point is called a singular point of the given function.

If every point of a given region $S$ is a regular point of a single valued function $f(Z)$, then $f(Z)$ is said to be holomorphic in $S$. We speak of a region as being a continuum of inner points, hence a region does not include its boundary, unless specified. We shall say that a function of $Z$ is analytic if it is holomorphic in at least some region $S$ with possible exception of certain singular points which do not interrupt the continuity of $S$.

From the definition of a function which is holomorphic in a region, we have at once the following general properties. Given two functions $f(Z)$ and $\phi(Z)$, each holomorphic in a region $S$; then it follows that in $S$:

1. $f(Z) + \phi(Z)$ is holomorphic.
2. $f(Z) \cdot \phi(Z)$ is holomorphic.
3. $f(Z)/\phi(Z)$ is holomorphic except for those values of $Z$ for which $\phi(Z) = 0$.
4. If $W_0$ is a regular point of $f(W)$ and $Z_0$ is a regular point of $W = \phi(Z)$, $\phi(Z_0) = W_0$ then $Z_0$ is a regular point of the function $f[\sqrt{\phi(Z)}]$, considered as a function of $Z$.

From the above properties, it follows that every rational integral function of $Z$ is an analytic function, holomorphic in
the finite region of the complex plane. Since every rational function is holomorphic, except at most a finite number of points where the denominator is zero, it also is an analytic function $f(z; 45)$. 
CHAPTER II

LINE-INTEGRALS

Definition of a Line Integral

In the calculus of a real variable, a definite integral is defined as the limit of a sum; that is

\[ \int_{a}^{b} f(x) \, dx = \lim_{N \to \infty} \sum_{k=1}^{N} f(x_k) \Delta x_k. \]

Stated in words: the portion of the x-axis between a and b is divided into N parts, the length \( \Delta x_k \) of each division is multiplied by the value of the function at some arbitrary point \( x_k \) in that division and the limit of the sum of those products is taken, as the number of such divisions simultaneously approach zero. \( \int \) 46-47. We must notice that the N divisions are taken along the x-axis and that axis is taken as the path of integration.

Let AB be one of the finite number of divisions of which an ordinary curve is composed. Let the coordinates

![Diagram](image)

Figure 1
of the points A and B be \((X_0, Y_0)\) and \((X_N, Y_N)\) respectively. Divide the arc AB into \(N\) parts by the insertion of \(N-1\) points \(P_1, P_2, \ldots, P_k, \ldots, P_{N-1}\), whose coordinates are \((X_1, Y_1), (X_2, Y_2), \ldots, (X_k, Y_k), \ldots, (X_{N-1}, Y_{N-1})\) respectively. Select at pleasure a point \((f_k, \eta_k)\) upon each arc \((P_{k-1}, P_k)\) where \(k = 1, 2, 3, \ldots, N\). Having given a function \(F(X, Y)\) which is continuous in \(X\) and \(Y\) together along the given curve, form the sum of the products of the subintervals

\[
\Delta_k X = X_k - X_{k-1}
\]

and the values of the given function \(F(X, Y)\) at the points \((f_k, \eta_k)\). Finally, consider the limit of this sum as the number of subintervals \(\Delta_k X\) between A and B is increased indefinitely, at the same time the lengths of these subintervals tend to zero. Thus we have the following limit

\[
\lim_{N \to \infty} \sum_{k=1}^{N} (f_k, \eta_k) \Delta_k X.
\]

Should the above limit exist, we define it as the line integral; where the integral was formed by taking the sum of the subintervals and letting them represent the desired integral.

In defining a line integral we took the limit of a sum of the products formed by multiplying \(F(f_k, \eta_k)\) by the
orthogonal projection of the arc \((P_{k-1}, P_k)\) upon the X-axis.

It may be easily observed that the ordinary definite integral is merely a special case of the line integral, where one of the coordinate axes is taken as the path of integration.

The existence of the limit defining a line integral may be made to depend upon that defining an ordinary definite integral \(\int_7^{48} \). That is to say, assume that \(F(X, Y)\) is a continuous function of the two variables \(X\) and \(Y\) together along the path of integration. Let \(Y = \phi(X)\) be a given curve \(AB\). Let this curve or arc be cut by a line in more than one point, say that the points are \(Y_1, Y_2\), \(\ldots\) as described by the figure.

\[\text{Figure 2}\]

In the above mentioned case the \(AB\) is composed of the following arcs \(AD\), \(DC\) and \(CB\) where each satisfies the necessary conditions. Thus we may write

\[
(2.3) \quad \int_{AB} F(X,Y) \, dx = \int_{AD} F(X,Y) \, dx + \int_{DC} F(X,Y) \, dx + \int_{CB} F(X,Y) \, dx.
\]

If \(P\), \(Q\) are two real functions of \(X\), \(Y\), we shall understand by
the line integral

\[
\begin{align*}
(2.4) \int_{X_0,Y_0}^{X_N,Y_N} P\,dx + Q\,dy \quad \text{or} \quad \int_C P\,dx + Q\,dy
\end{align*}
\]

to be the sum of the two line integrals

\[
\begin{align*}
\int_{X_0,Y_0}^{X_N,Y_N} P\,dx, \quad \int_{X_0,Y_0}^{X_N,Y_N} Q\,dy
\end{align*}
\]

From our definition we know that these integrals exist because they were assumed or given to be continuous.

**Plane Area as a Line Integral**

In using line integrals around closed curves we need some method of distinguishing between the two directions in which the curves may be traversed. Accordingly, when the curve is part of the boundary of a specified region, we shall say that a positive direction is that in which a person walking around the curve has the region on his left-hand side. Thus, if a circle is the boundary of the region included within it, the positive direction is counterclockwise, but if the same circle is a part of the boundary of a region exterior to it, the positive direction is now clockwise \( \lceil 8; 177 \rceil \).
Let us consider the integral

\[ \int_C y \, dx \]

taken in the positive direction along a closed curve \( C \), which bounds a region of area.

For simplicity, let us assume first that \( C \) is such that a line parallel to \( OX \) or \( OY \) meets it in two points or not at all, with the exception of four tangents which are parallel to \( OX \) or \( OY \) at points \( L, K, M \) and \( N \). Here \( L \) is the extreme left-hand point of the curve, \( K \) the extreme right-hand point, \( M \) the highest point and \( N \) the lowest point - Let us draw the tangents \( FL, \)
BK, DM, and EN. The integral taken along C from L through N to K gives the area FLNKB. The integral taken along C from K through M to L gives in magnitude the area FLNKB but the sign is negative since dX is always negative.

\[(2.6) \quad \text{The area } FLNKB = \int_{FL} f(x,y) \, dx + \int_{LN} f(x,y) \, dx + \int_{NK} f(x,y) \, dx + \int_{KB} f(x,y) \, dx.\]

\[
\text{The area } FLNKB = \int_{FL} f(x,y) \, dx + \int_{LM} f(x,y) \, dx + \int_{MK} f(x,y) \, dx + \int_{KB} f(x,y) \, dx.
\]

Therefore the sum of these two directions around C gives us the sum of the area, namely the area bounded by C with the negative sign, i.e. area is

\[(2.7) \quad A = - \int_{C} x \, dx.\]

Consider in a similar manner the integral

\[(2.8) \quad \int_{C} x \, dy.\]

The above integral taken along C from N through K to M gives the area LMKMD. The integral taken along C from M through L to N
gives the area $\text{SNMD}$ with a negative sign.

\begin{equation}
(2.9) \quad \text{The area } \text{ENKMD} = \int_{en} f(X,Y)\,dY + \int_{nk} f(X,Y)\,dY + \int_{km} f(X,Y)\,dY + \int_{md} f(X,Y)\,dY.
\end{equation}

\begin{equation}
\text{The area } \text{ENMLD} = \int_{en} f(X,Y)\,dY + \int_{nl} f(X,Y)\,dY + \int_{lm} f(X,Y)\,dY + \int_{md} f(X,Y)\,dY.
\end{equation}

Taking the sum of the above areas we get

\begin{equation}
(2.10) \quad A = \int_{C} X\,dY.
\end{equation}

By adding the two areas, $A = -\int_{C} Y\,dX$ and $A = \int_{C} X\,dY$ we get

\begin{equation}
(2.11) \quad 2A = \int_{C} (X\,dY - Y\,dX) \quad \text{or} \quad A = \frac{1}{2} \int_{C} (X\,dY - Y\,dX)
\end{equation}

which expresses the area in terms of a line integral taken around the boundary of the area.

Green's Theorem

One of the important theorems associated with line integrals gives, under certain conditions, a relation between such
integrals and ordinary double integrals. This gives us a way of expressing an ordinary integral as a function of two real variables.

Green's theorem may be stated as follows: In a given finite region $S$ let $C$ be the complete boundary of any portion of the plane such that $C$ lies within $S$ and incloses only points of $S$. If in a given region $P(X,Y)$ and $Q(X,Y)$ are continuous real functions of $X$ and $Y$ together, having the continuous partial derivatives $\frac{\partial Q}{\partial X}, \frac{\partial P}{\partial Y}$ then

\[
(212) \int_C (Pdx + QdY) = \iint_R - \frac{\partial Q}{\partial X} - \frac{\partial P}{\partial Y} dxdy
\]

where the double integral is to be taken over the region bounded by $C$.

Let $F(X,Y)$ be any function which is continuous in $R$ and on $C$ for which $\frac{\partial F}{\partial Y}$ is continuous.

![Figure 5](image)

We shall consider the double integral of $\frac{\partial F}{\partial Y}$ over the area $R$. Then we may evaluate the multiple integral thusly.
\[ (2.13) \quad \iint_R \frac{\partial P}{\partial Y} \, dX dY = \int_a^b \int_{Y_2}^{Y_1} \frac{\partial P}{\partial Y} \, dY \, dX \]

\[ = \int_a^b \left[ P(X,Y_2) - P(X,Y_1) \right] \, dX \]

\[ = -\int_a^b P(X,Y_1) \, dX - \int_b^a P(X,Y_2) \, dX \]

But by our definition of a line integral, the expression

\[ -\int_a^b P(X,Y_1) \, dX - \int_b^a P(X,Y_2) \, dX \]

is the line integral of \( PdX \) around \( C \) in the positive direction. Hence we have

\[ \iint_R \frac{\partial P}{\partial Y} \, dX dY = -\int_c PdX \]

where the indices \( R \) and \( C \) are used to denote the region and the curve over which the integrals are taken, and where the direction along \( C \) is to be positive.

Similarly, if \( Q \) is another function of \( X \) and \( Y \), continuous in \( R \) and on \( C \) and such that \( \frac{\partial Q}{\partial X} \) is continuous in \( R \), we may show that

\[ (2.14) \quad \iint_R \frac{\partial Q}{\partial X} \, dX dY = \int_c QdX \]
We now have

\[ \iint_R \frac{\partial P}{\partial Y} \, dx \, dy = \oint_C P \, dx \quad \text{and} \quad \iint_R \frac{\partial Q}{\partial X} \, dx \, dy = \oint_C Q \, dy. \]

By subtracting the last integral from the first we get

\[ (2.15) \iint_R \left( \frac{-\partial P}{\partial Y} - \frac{\partial Q}{\partial X} \right) \, dx \, dy = -\oint_C (P \, dx + Q \, dy) \]

which establishes Green's theorem.
CHAPTER III

CAUCHY'S FIRST INTEGRAL THEOREM

Let \( f(Z) \) be an analytic function, regular in a domain \( D \). Let \( Z_0 \) and \( Z_1 \) be two points of \( D \), joined by an arc \( L \), every point of which belongs to the domain \( D \). Then the integral of 
\( f(Z) \) along \( L \) certainly exists since \( f(Z) \) is continuous on \( L \). The fundamental property on which the theory of analytic functions depends, is that the value of this integral is a function of \( Z_0 \) and \( Z_1 \) alone and is quite independent of the arc which joins the two given points \( Z_0 \) and \( Z_1 \).

An equivalent form of this result is Cauchy’s Theorem, which states that, if \( C \) is a simple closed curve lying in \( D \), then the integral of \( f(Z) \) around \( C \) is zero. For any two points of \( C \), divide it into two curves \( L \) and \( L_1 \), let us say. If \( L_2 \) denotes the curve \( L_1 \) described in the opposite sense, the integral of \( f(Z) \) along \( L \) and \( L_2 \) are equal. Hence we have

\[
(3.1) \quad \int_C f(Z) dZ = \int_L f(Z) dZ + \int_{L_1} f(Z) dZ = \int_L f(Z) dZ - \int_{L_2} f(Z) dZ = 0.
\]

It is very difficult to prove Cauchy's Theorem in its general form, but here we shall prove it from an elementary standpoint.

The Elementary Proof of Cauchy's Theorem

We now prove the simplest form of Cauchy's theorem, that,
if $f(Z)$ is an analytic function whose derivative $f'(Z)$ exists and is continuous at each point within and on a closed curve, then

$$\left(3.2\right) \oint f(Z)\,dz = 0.$$  

Let $D$ be the closed domain which consists of all points within and on $C$. If we write $Z = X + iY$ and $f(Z) = U + iV$ where $U$ and $V$ are real functions of the real variables $X$ and $Y$, we have

$$\left(3.3\right) \oint f(Z)\,dz = \oint (U\,dx - V\,dy) + i\oint (V\,dx + U\,dy)$$

where by $\oint (P\,dx + Q\,dy)$ we mean each of these integrals is obtained by means of Green's theorem which states that if $P(X,Y), Q(X,Y), \frac{\partial Q}{\partial X}, \frac{\partial P}{\partial Y}$ are continuous functions of both variables in $D$, then

$$\left(3.4\right) \oint (P\,dx + Q\,dy) = \iint_D \left( \frac{\partial Q}{\partial X} - \frac{\partial P}{\partial Y} \right)\,dxdy$$

By hypothesis $f'(Z)$ exists and is continuous in $D$. Since, however,

$$\left(3.5\right) f'(Z) = U_X + iV_X = V_Y - iU_Y .$$

We are in each assuming that $U$ and $V$ and their partials are continuous functions of both variables $X$ and $Y$ in $D$. The
condition of Green's theorem satisfied, we see that

$$\int_C f(z)dz = -\iint_D \left( \frac{\partial V}{\partial x} + \frac{\partial U}{\partial y} \right) dxdy + \iint_D \frac{\partial U}{\partial x} dxdy$$

$$- \frac{\partial V}{\partial y} dxdy = 0.$$  

The General Form of Cauchy's Theorem

Let $f(z)$ be holomorphic in a region $S$ and let $C$ be the complete boundary of any portion $S'$ of $S$ such that $C$ lies wholly in $S$ and incloses only points of $S$; then

$$\int_C f(z)dz = 0.$$  

In order to prove this theorem in its general form, we must establish a lemma. Let us consider the case where $C$ is a single closed ordinary curve and the inclosed region $S'$ is simply connected. Let $S'$ be divided into squares and partial squares;

![Figure 6](image)

let $N$ denote the number of squares of $S_1$ and $m$ the number of partial squares of $R_1$. If the integral is taken in a positive
direction around the edges of the various sub-regions $S_i, R_i$, it will be seen that each side of these regions that is not a portion of $C$ is taken twice as a path of integration, the two integrals being in opposite directions. Considering the sum of the integrals about the edges of all the regions $S_i, R_i$, we may therefore write

$$\int_C f(Z)\,dZ = \sum_{i=1}^{n} \int_{\nu_i} f(Z)\,dZ + \sum_{i=1}^{n} \int_{\lambda_i} f(Z)\,dZ$$

where $\nu_i, \lambda_i$ denotes the boundaries of $S_i, R_i$ respectively.

From the lemma, we have within or upon the boundary of each $S_i, R_i$ a point $Z_1$ such that

$$\left| \frac{f(Z) - f(Z_1)}{Z - Z_1} - f'(Z) \right| < \varepsilon.$$  

This relation can be written in the form

$$f(Z) - f(Z_1) = (Z - Z_1)f'(Z) + \eta_i(Z - Z_1)$$

where $\eta_i$ is a function of $Z$ such that $|\eta_i| < \varepsilon$

when $Z$ varies along the contour of $S_i$ or $R_i$. Let us now consider the integral around the perimeter of one of the squares $S_i$. We have
\begin{equation}
\int_{\gamma_i} f(Z) dZ = \int_{\gamma_i} f(Z_i - Z_i f'(Z_i)) f'(Z_i) Z dZ + \int_{\gamma_i} \eta_i(Z - Z_i) dZ.
\end{equation}

But we know that the following conditions are true
\begin{equation}
\int_{d \gamma} ^{\beta} dZ = \beta - \eta, \quad \int_{d \eta} ^{\beta} Z dZ = \frac{1}{2} (\beta^2 - \eta^2).
\end{equation}

In the particular case under consideration, as the path of integration is on a closed curve, \( \beta \) and \( \eta \) are both the same point, hence both of the integrals in (3.10) vanish. Hence
\begin{equation}
\left| \int_{\gamma_i} f(Z) dZ \right| = \left| \int_{\gamma_i} \eta_i(Z - Z_i) dZ \right|.
\end{equation}

Let the length of one side of the square \( S_i \) be \( C_i \). The diagonal of the square is \( C_i \sqrt{2} \). Hence we have
\[ |Z - Z_i| \leq C_i \sqrt{2}. \]

Consider the integral taken around one of the partial squares \( \gamma_i \), we have
\begin{equation}
\int_{\gamma_i} f(Z) dZ = \int_{\gamma_i} f(Z_i) - Z_i f(Z) f'(Z) Z dZ + f'(Z) \int_{\gamma_i} Z dZ + \int_{\gamma_i} \eta_i(Z - Z_i) dZ.
\end{equation}

As we have shown that the first two integrals in the second
member of the above equation vanishes, we have

\[ (3.13) \quad \left| \int_{\mathcal{A}} f(z) \, dz \right| = \left| \int_{\mathcal{A}_1} \eta(z - z_1) \, dz \right|. \]

We denote by $C_i$ the length of a side of the square of which $R_i$ is a portion; $R_i$ being that portion of the square cut off by the curve $C$ and lying in $S'$. Let $l_i$ be the length of that arc of $C$ which forms a portion of the boundary of $R_i$. We have then

\[ (3.14) \quad \left| \int_{\mathcal{A}} f(z) \, dz \right| \leq C_i \sqrt{2} \quad \text{and thus} \quad \left| \int_{\mathcal{A}_1} \eta(z - z_1) \, dz \right| \leq \sqrt{2} (4C_i + l_i) \]

\[ \leq \varepsilon \sqrt{2} (4B_i + C_i l) \]

where $B_i$ denotes the area of the square of which $R_i$ is a part, and $C_i$ the length of one side of the largest of the squares that come into consideration in the sub-division of $S' \backslash 3; 68 \}.$

Replacing each form of the sum in our definition by its absolute value, we have the following condition

\[ \left| \int_{\mathcal{C}} f(z) \, dz \right| < \varepsilon \sqrt{2} \left( \sum_{i=1}^{n} 4A_i + \sum_{i=1}^{n} (4B_i + C_i l) \right) \leq \sqrt{2} (4A + CL) \]

where we say $L$ denotes the length of the curve $C$ and $A$ denotes the combined area of our squares with which the region $S'$ was originally composed. The expression included in the braces is therefore a constant and as $\varepsilon$ is arbitrarily small, the product is arbitrarily small. As the absolute value of the integral

\[ \int_{\mathcal{C}} f(z) \, dz \]

is shown to be less than an arbitrarily small
number, it follow that

\[ \int_{C} f(z)dz = 0, \]

which proves our theorem.
Development of Cauchy's Second Integral Theorem

Consider a finite closed region $S$ whose boundary $C$ consists of a finite number of ordinary curves. If $f(Z)$ is holomorphic within $S$ and converges uniformly along $C$ or if it is also holomorphic for values along $C$, then for any inner point $a$ of $S$ we have $\int_{7; 75}^{\infty}$.

\[(4.1) \quad f(a) = \frac{1}{2 \pi i} \int \frac{f(Z) dZ}{Z - a} \]

Let $f(Z)$ be holomorphic in the finite region $S'$ limited by the boundary $C$, composed of one of several distinct closed curves, and continuous on the boundary itself. If $a$ is a point of the region $S'$, the function is holomorphic in the same region except at the point $Z = a$.

With the point $a$ as center, let us describe a circle with radius $\rho$ lying entirely in the region $S'$, the preceding
function is holomorphic in the region of the plane limited by the boundary C and the circle \( \gamma_1 \) and we can apply Cauchy's first theorem to it. Suppose, for definiteness, that the boundary C is composed of two closed curves \( C_1 \) and \( C_2 \) (figure 7), then we have

\[
(4.2) \quad \int_C \frac{f(Z)\,dZ}{Z-\alpha} = \int_{C_1} \frac{f(Z)\,dZ}{Z-\alpha} + \int_{C_2} \frac{f(Z)\,dZ}{Z-\alpha}
\]

where the three integrals are taken in the same sense, we can write this in the following form

\[
(4.3) \quad \int_{C_1} \frac{f(Z)\,dZ}{Z-\alpha} = \int_{C_2} \frac{f(Z)\,dZ}{Z-\alpha}
\]

where the integral around the curve denotes the integral taken along the total boundary C in the positive sense. If the radius \( \rho \) of the circle \( \gamma \) (figure 7) is very small, the value of \( f(Z) \) at any point of this circle differs very little from \( f(\alpha) \), that is \( f(Z) \approx f(\alpha) + \delta \) where \( |\delta| \) is very small. Replacing \( f(Z) \) by this value we find

\[
(4.4) \quad \int_C \frac{f(Z)\,dZ}{Z-\alpha} \approx f(\alpha) \int_C \frac{dZ}{Z-\alpha} + \int_C \frac{\delta\,dZ}{Z-\alpha}
\]

The first integral on the right-hand side is easily evaluated; if we put \( Z-\alpha = \rho e^{i\theta} \) or \( Z = \alpha + \rho e^{i\theta} \), it becomes
\[(4.5) \int_{C} \frac{dZ}{Z - \alpha} = \int_{0}^{2 \pi i} i e^{i \theta} d\theta = 2 \pi i. \]

The second integral \( \int_{C_{1}} \frac{dZ}{Z - \alpha} \) is therefore independent of the radius \( \rho \) of the circle; on the other hand, if \( |\sigma| \) remains less than a positive number \( \rho \), the absolute value of this integral is less than \( (\rho / \rho)^2 \pi = 2 \pi \rho \). Now since the function \( f(Z) \) is continuous for \( Z = \alpha \), we can choose the radius so small that \( \rho \) also will be as small as we wish. Hence this integral must be zero. Dividing the two sides of the equation (4.4) by \( 2 \pi i \), we get

\[(4.6) \quad f(\alpha) = \frac{1}{2 \pi i} \int_{C} \frac{f(Z) dZ}{Z - \alpha}. \]

This is Cauchy’s fundamental formula. It expresses the value of the function \( f(Z) \) at any point \( \alpha \) whatever within the boundary by means of the values of the same function taken along that boundary \[3; 76].

**Derivative of an Analytic Function**

For a given function to be analytic in a region, the derivative of the function must exist at every point, except at most a set of points which do not interrupt the continuity of the region.
Let \( q + \Delta q \) be any point near \( q \), which for example, we suppose lies in the interior of the circle \( C \) (figure 7) of radius \( \rho \). Then we have also

\[
(4.7) \quad f(q + \Delta q) = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{z - (q + \Delta q)}
\]

and consequently by our definition of differentiation, we get by subtracting (4.6) from (4.7)

\[
(4.8) \quad \Delta q \rightarrow 0 \quad \frac{f(q + \Delta q) - f(q)}{\Delta q} = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - q)(z - (q + \Delta q))}
\]

\[
= \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - q)^2}
\]

and consequently it appears that

\[
f'(q) = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - q)^2}.
\]

In order to prove rigorously that we have the right to apply the usual formula for differentiation, let us write the integral in the form

\[
\int_C \frac{f(z)dz}{(z - q)(z - (q + \Delta q))} = \int_C \frac{f(z)dz}{(z - q)^2} + \int_C \frac{\Delta q f(z)dz}{(z - q)^2(z - (q + \Delta q))}.
\]

Let \( M \) be an upper bound for \( |f(z)| \) along \( C \). Let \( L \) be the length of the boundary and \( S \) the lower bound for the distance of any
point whatever of the circle \( Y \) to any point \( C \). The absolute value of the last integral is less than \( ML(\Delta \zeta) / \epsilon^3 \) and consequently approaches zero with \( |\Delta \zeta| \). Passing to the limit, we obtain the following results

\[
f'(\zeta) = \frac{1}{2 \pi i} \int_C \frac{f(z)dz}{(z - \zeta)^2}.
\]

A condition for differentiating under the integral sign is for the given function to be continuous within a certain region or interval \( \overline{3; 141.7} \). Our function was given to be continuous; therefore, by applying our definition or conditions for differentiation under the integral sign we may obtain the following:

\[
f''(\zeta) = \frac{2!}{2 \pi i} \int_C \frac{f(z)dz}{(z - \zeta)^3},
\]

\[
f'''(\zeta) = \frac{3!}{2 \pi i} \int_C \frac{f(z)dz}{(z - \zeta)^4},
\]

\[
f^N(\zeta) = \frac{N!}{2 \pi i} \int_C \frac{f(z)dz}{(z - \zeta)^{N+1}}.
\]

Hence if a function \( f(z) \) is analytic in a certain region of the plane, the sequence of successive derivatives of that function is unlimited, and all these derivatives are also analytic functions in the same region \( \overline{3; 141.7} \).
CHAPTER V

POLES AND RESIDUES

Poles

Every function analytic in a circle with center $a$ is equal, in the interior of that circle, to the sum of a power series

$$f(z) = A_0 + A_1(z - a) + \ldots + A_m(z - a)^m + \ldots$$

We shall say, for brevity, that the function is regular at the point $a$, or that $a$ is a regular point for the given function. We shall call the interior of a circle $C$, described about $a$ as a center with the radius $\rho$, the neighborhood of the point $a$, when the series (5.1) is applicable. It is, moreover, not necessary that this shall be the largest circle in the interior of which the series (5.1) is true. Radius $\rho$ of the neighborhood will often be defined by some other particular property.

If the first coefficient $A_0$ is zero, we have $f(a) = 0$, and the point $a$ is a zero of the function $f(z)$. The order of a zero is defined in the same way as for polynomials. If the development of $f(z)$ commences with a term of degree $m$ in $z - a$,

$$f(z) = A_m(z - a)^m + A_{m+1}(z - a)^{m+1} + \ldots (m > 0)$$

where $A_m \neq 0$, we have

$$f(a) = 0, f'(a) = 0, \ldots, f^{m-1}(a) = 0, f^m(a) \neq 0$$
and the point \( a \) is said to be a zero of order \( m \). We can also write the preceding series in the form

\[
(5.3) \quad f(Z) = (Z - a)^m g(Z),
\]

\( g(Z) \) being a power series which does not vanish when \( Z = a \). Since this series is a continuous function of \( Z \), we can choose the radius \( r \) of the neighborhood so small that \( g(Z) \) does not vanish in that neighborhood, and we see that the function \( f(Z) \) will not have any other zero than the point \( a \) in the interior of that neighborhood. The zeros of an analytic function are therefore isolated points.

Every point which is not a regular point for a single-valued function \( f(Z) \) is said to be a singular point. A singular point of the function \( f(Z) \) is a pole if that point is a regular point for the reciprocal function \( 1/f(Z) \). The development of \( 1/f(Z) \) in powers of \( Z - a \) cannot contain a constant term, for the point \( a \) would then be a regular point for the function \( f(Z) \).

Let us suppose that the development commences with a term of degree \( m \) in \( Z - a \),

\[
(5.4) \quad 1/f(Z) = (Z - a)^m g(Z),
\]

where \( g(Z) \) denotes a regular function in the neighborhood of the point \( a \) which is not zero when \( Z = a \). From this we derive

\[
(5.5) \quad f(Z) = \frac{1}{(Z - a)^m g(Z)} = \frac{g(Z)}{(Z - a)^m}.
\]

where \( g(Z) \) denotes a regular function in the neighborhood of
the point \( q \) which is not zero when \( Z = q \). We may write (5.5) in its equivalent form

\[
(5.6) \quad f(Z) = \sum_{n=0}^{m} \frac{B_n}{(Z - q)^n} + \frac{B_{m-1}}{(Z - q)^{m-1}} + \ldots + \frac{B_1}{Z - q} + P(Z - q),
\]

where we denote by \( P(Z - q) \), as we did for \( \phi(Z) \) in (5.4), a regular function for \( Z = q \), and by \( B_m, B_{m-1}, \ldots, B_1 \) certain constants. Some of the coefficients \( B_1, B_2, \ldots, B_{m-1} \) may be zero but \( B_m \) is surely different from zero. The integer \( m \) is called the order of the pole. Thus we say that we have a pole if the denominator goes to zero for \( Z = q \) and the numerator is different from zero.

Residues

Cauchy gave us an important theorem on residues, which is stated below.

Let \( f(Z) \) be continuous within and on a closed contour and regular, save for a finite number of within \( C \). Then

\[
(5.7) \quad \oint_C f(Z) dZ = 2\pi i \sum \text{sum of the residues of } f(Z) \text{ at its poles in } C.
\]

Figure 8
If we denote by $Z_1, Z_2, \ldots, Z_n$ the poles of $f(Z)$ within $C$, we can evidently draw a set of circles $C_r$, of radius $\rho$ and center $Z_r$, which do not intersect and lie within $C$, provided that $\rho$ is sufficiently small. Then $f(Z)$ is regular in the domains bounded externally by the circle $C$ and internally by the circles $C_r$. We can therefore deform $C$ continuously without crossing a singularity of $f(Z)$ until it consists of the circles $C_r$ joined by the polygon $P$ as shown in the preceding figure. We have

\[(5.8) \int_C f(Z)\,dZ = \int_{\rho} f(Z)\,dZ + \sum_{r=1}^{n} \int_{C_r} f(Z)\,dZ = \sum_{r=1}^{n} \int_{C_r} f(Z)\,dZ.\]

The integral around the polygon $P$ vanishes since it is regular within. Let $Z_v$ be a pole of order $m$, such that

\[(5.9) f(Z) = \phi(Z) + \sum_{s=1}^{m} \frac{q_s}{(Z - Z_v)^s}\]

when $\phi(Z)$ is regular within and on $C_r$. Then

\[(5.10) f(Z)\,dZ = \sum_{s=1}^{m} \int_{C_r} \frac{q_s\,dZ}{(Z - Z_v)^s}.\]

Now on $C_r$, $Z = Z_r + \epsilon e^{i\phi}$ where $\phi$ varies from 0 to $2\pi$.
Making this substitution we get

\[(5.11) \int_{C_r} f(Z) dZ = \sum_{s=1}^{n} q_s e^{i-\sigma} \int_{0}^{2\pi} e^{(1-s)i} \phi d\phi = 2\pi i \phi \]

which equals \(2\pi i\) (residue of \(f(Z)\) at \(Z_r\)).

From the foregoing statement (5.11), it follows that

\[\int_{C} f(Z) dZ = \sum_{r-1}^{n} \int_{C_r} f(Z) dZ = 2\pi i \sum (\text{residues of } f(Z) \text{ at } Z_r)\]

which proves Cauchy’s theorem on residues.

The Residue from Laurent’s Expansion

If \(f(Z)\) is holomorphic in a given finite region \(S\) except at \(Z_0\), where it has a pole, then the residue of \(f(Z)\) at \(Z_0\) is equal to the coefficient of \((Z - Z_0)^{-1}\) in the expansion of \(f(Z)\) in powers of \((Z - Z_0)^{-1}\) of the form

\[(5.12) \quad f(Z) = \frac{q-h}{(Z - Z_0)^h} + \frac{q-h+1}{(Z - Z_0)^{h-1}} + \cdots + \frac{q-h}{(Z - Z_0)^1} + q_0 + \cdots\]
\[ f(Z) = \frac{d-h}{(Z-Z_0)^h} + \frac{d-h+1}{(Z-Z_0)^{h-1}} + \ldots + \frac{d-1}{Z-Z_0} + \phi(Z) \]

where \( \phi(Z) \) is a function that is holomorphic in the neighborhood of \( Z_0 \); say within and upon a circle \( C \) having \( Z_0 \) as a center.

Taking \( C \) as the path of integration, we have

\[
(5.13) \quad \int_C f(Z) dZ = \cdot h \int_C (Z-Z_0)^{-h} dZ + \ldots + \cdot 1 \int_C (Z-Z_0)^{-1} dZ
\]

\[ + \int_C \phi(Z) dZ \]

where the integral \( \int_C \phi(Z) dZ \) vanishes, since \( \phi(Z) \) is holomorphic in the closed region bounded by \( C \). To calculate the remaining integrals we use the relation

\[
\int_C (Z-Z_0)^n dZ = \begin{cases} 
0 & n \neq -1, \\
\frac{2\pi i}{n} & n = -1.
\end{cases}
\]

Therefore we have \( \int_C f(Z) dZ = 2\pi i \cdot \cdot 1 \) which gives the residue of \( f(Z) \) at \( Z_0 \).

The Residue for \( f(Z) = \frac{P(Z)}{Q(Z)} \)

The importance of residues is now evident, and it is useful
to know how to calculate them rapidly. If a point \( \alpha \) is a pole of order \( m \) for \( f(Z) \), the product \( (Z - \alpha)^m f(Z) \) is regular at the point \( \alpha \), and the residue of \( f(Z) \) is evidently the coefficient of \( (Z - \alpha)^{m-1} \) in the development of that product. The rule becomes simple in the case of a simple pole; the residue is then equal to the limit of the product \( (Z - \alpha) f(Z) \) for \( Z = \alpha \). Quite frequently the function \( f(Z) \) appears under the form

\[
\frac{f(Z)}{Q(Z)} = \frac{P(Z)}{Q(Z)}
\]

where the function \( P(Z) \) and \( Q(Z) \) are regular for \( Z = \alpha \), and \( P(\alpha) \) is different from zero while \( Q(\alpha) \) is a simple zero of \( Q(Z) \). Let \( Q(Z) = (Z - \alpha)R(Z) \); then the residue is equal to the quotient \( \frac{P(\alpha)}{R(\alpha)} \), or it is equal to

\[
\frac{P(\alpha)}{Q'(\alpha)}
\].
CHAPTER VI

SOLUTIONS OF REAL INTEGRALS

Evaluation of Definite Integrals

Consider a function \( F(Z) \); with the origin as center let us describe a circle \( C \) with a radius \( R \) large enough to include all the roots of the denominator of \( F(Z) \), and let us consider a path of integration formed by the diameter \( BA \), traced along the real axis, and the semi-circumference \( C' \), lying above the real axis. The only singular points of \( F(Z) \) lying interior of this path are poles, which come from the roots of the denominator of \( F(Z) \) for which the coefficients of \( i \) are positive. Indicating by \( \sum \frac{R_k}{k} \) the sum of the residues relative to these poles we can then write

\[
\left(6.1\right) \int_{-R}^{R} F(Z) dZ + \int_{C'} F(Z) dZ = 2 \pi i \sum \frac{R_k}{k}.
\]

The definite integral \( \int_{-\infty}^{\infty} F(X) dX \), taken along the real axis, where \( F(X) \) is a rational function, has a sense, provided the denominator does not vanish for any real value of \( X \) and the degree of the numerator is less than the degree of the denominator or by at least two. This must be true because we do not want the numerator to approach infinity as rapidly as our denominator for our rational function would not have meaning if it did.

This being true, as the radius \( R \) becomes infinite the integral
along C', in (6.1), approaches zero. Since the product $ZF(Z)$ is zero for $Z$ infinite, taking the limit, we get

\[ 3; 96 \]

\[
(6.2) \quad \int_{-\infty}^{\infty} F(x)dx = 2 \pi i \sum R_n .
\]

If $F$ is a rational function of $\sin X$ and $\cos X$ that does not become infinite for any real value of $X$, and where the integral is to be taken along the axis of reals, we can reduce (6.2) to the case where

\[
\int_{0}^{2 \pi} F(\cos X, \sin X)dx
\]

is a definite integral.

Let us first notice that we do not change the value of the integral by taking for the limits $X_0$ and $X_0 + 2 \pi$, where $X_0$ is any real number whatever. It follows that we can take for the limits $-\pi$ to $\pi$, for example. Now the classic change of variables $\tan (X/2) = t$ reduces the given integral to the integral of a rational function of $t$ taken between the limits $-\infty$ and $+\infty$, for $\tan X/2$ increases from $-\infty$ to $+\infty$ when $X$ increases from $-\pi$ to $\pi$.

Example 1. - Find \[ \int_{0}^{\infty} \frac{\sin X}{X} dx \] by considering \[ \int_{C} \frac{e^{iZ}}{Z} dZ \]

around $B'A'$. The function $e^{iZ/Z}$ is holomorphic in the interior of the boundary formed by the two semi-circles $B'MO$, $A'NA$ described about the origin as center with the radii $R$ and $r$, and the straight lines $AB$, $B'A'$ we have,
then, the relation
\[\int_{r}^{R} \frac{e^{ix}}{x} \, dx + \int_{E}^{B} \frac{e^{iz}}{z} \, dz + \int_{-R}^{r} \frac{e^{ix}}{x} \, dx + \int_{A}^{A'} \frac{e^{iz}}{z} \, dz = 0\]
which can be written in the form
\[\int_{r}^{R} \frac{e^{ix} - e^{-ix}}{x} \, dx + \int_{E}^{B} \frac{e^{iz}}{z} \, dz + \int_{A}^{A'} \frac{e^{iz}}{z} \, dz = 0.\]

When \(r\) approaches zero, the last integral approaches \(-\pi i\); we have, in fact,
\[\frac{e^{iz}}{z} = \frac{1}{z} + P(Z)\]
where \(P(Z)\) is a regular function at the origin, so that
\[\int_{A}^{A'} \frac{e^{iz}}{z} \, dz = \int_{A}^{A'} \frac{dZ}{z} + \int_{A}^{A'} P(Z) \, dZ.\]
The integral of the regular part \(P(Z)\) becomes infinitesimal with the length of the path of integration; as for the first integral, it is equal to the variation of \(\log (Z)\) along \(A'NA\), that is, to \(-\pi i\).

The integral along \(E_{B}B'\) approaches zero as \(R\) becomes infinite.

For if we put \(Z = R(e^{i\theta} + i \sin \theta)\) we find
\[\int_{E_{B}B'} \frac{e^{iz}}{z} \, dz = i \int_{0}^{\pi} e^{-R \sin \theta + i R \cos \theta} \, d\theta\]
and the absolute value of this integral is less than
\[ \int_0^{\pi/2} e^{-R \sin \theta} \, d\theta = 2 \int_0^{\pi/2} e^{-R \sin \theta} \, d\theta. \]

When \( \theta \) increases from 0 to \( \pi/2 \), the quotient \( \frac{\sin \theta}{\theta} \) decreases from 1 to \( 2/\pi \), and we have \( R \sin \theta \geq \frac{2R}{\pi} \) hence
\[ e^{-R \sin \theta} < e^{-\frac{2R}{\pi}}. \]

\[ \int_0^{\pi/2} e^{-R \sin \theta} \, d\theta < \int_0^{\pi/2} e^{-\frac{2R}{\pi}} \, d\theta = -\frac{\pi}{2R} \left[ e^{-\frac{2R}{\pi}} \right]_0^{\pi/2} \]
\[ = \frac{\pi}{2R} \left[ 1 - e^{-\frac{R}{\pi}} \right]. \]

which gives, when we take the limit
\[ \int_0^\infty \frac{e^{iX} - e^{-iX}}{X} \, dX = \frac{\pi}{2} i. \] But should we take
\[ e^{iX} = \cos X + i \sin X; \quad e^{-iX} = \cos X - i \sin X, \]
combining the two we get \( 2i \sin X \), hence we can write the above results as
\[ 2i \int_0^\infty \frac{\sin X}{X} \, dX = \frac{\pi}{2} i \text{ or } \int_0^\infty \frac{\sin X}{X} \, dX = \frac{\pi}{2}. \]

Example 2. Find \( \int_0^\infty \cos X^2 \, dX \), \( \int_0^\infty \sin X^2 \, dX \) by considering the integral of the transcendental function \( e^{-z^2} \) along the
boundary OABO formed by the two radii OA and OB, making an angle of \(45^\circ\) and by the arc of a circle AB

\[
\int_0^R e^{-\frac{x^2}{2}} \, dx + \int_{\text{AB}} e^{-\frac{Z^2}{2}} \, dZ = \int_{\text{OB}} e^{-\frac{Z^2}{2}} \, dZ
\]

is equal to zero, and can be expressed as follows

when the radius \(R\) of the circle to which the arc AB belongs becomes infinite, the integral along the arc AB approaches zero. In fact if we put \(Z = R e^{\frac{\cos \theta}{2}} + i \sin \frac{\theta}{2} \), that integral becomes

\[
\frac{iR}{2} \int_0^{\frac{\pi}{2}} e^{-\frac{R^2}{2} \cos \theta} e^{i \frac{\theta}{2}} \, d\theta
\]

and its absolute value is less than the integral

\[
\frac{R}{2} \int_0^{\frac{\pi}{2}} e^{-\frac{R^2}{2} \cos \theta} \, d\theta. \quad \text{As in the previous example we have}
\]

\[
\frac{R}{2} \int_0^{\frac{\pi}{2}} e^{-\frac{R^2}{2} \cos \theta} \, d\theta = \frac{R}{2} \int_0^{\frac{\pi}{2}} e^{-\frac{R^2}{2} \sin \theta} \, d\theta < \frac{R}{2} \int_0^{\frac{\pi}{2}} e^{-\frac{R^2}{2}} \, d\theta.
\]
The last integral has the value

\[- \frac{11}{4R} \int_0^{2R/2} e^\frac{11/2}{2} \, d\rho = \frac{11}{4R} (1 - e^{-R^2})\]

and approaches zero when \( R \) becomes infinite.

Along the radius \( CB \) we can put \( Z = \rho (\cos 11/4 + i \sin 11/4) \)

which gives \( e^{-Z^2} = e^{-\rho^2} \) and as \( R \) becomes infinite we have

as the limit

\[\int_0^\infty e^{i\rho^2} (\cos 11/4 + i \sin 11/4) \, d\rho = \int_0^\infty e^{-x^2} \, dx = \sqrt{\frac{\pi}{2}}\]

or

\[\int_0^\infty e^{i\rho^2} \, d\rho = \sqrt{\frac{\pi}{2}} (\cos 11/4 - i \sin 11/4) .\]

Equating the real parts and the coefficients of \( i \) we get the values

\[\int_0^\infty \cos \rho^2 \, d\rho = \frac{1}{2} \sqrt{\frac{\pi}{2}} , \quad \int_0^\infty \sin \rho^2 \, d\rho = \frac{1}{2} \sqrt{\frac{\pi}{2}} .\]

Example 3. - Find \( \int_0^\infty \frac{\cos mx \, dx}{1 + x^2} \) by considering \( \int \frac{e^{imz} \, dz}{1 + z^2} \)

\((m > 0)\) taken along the closed path formed by the axis of reals from \(-R\) to \(R\) and a semi-circle from \(R\) back to \(-R\).
Within this path the function integrated has a pole at $Z = i$, for $i^2 = -1$ which makes the denominator zero. To find its residue we write

$$\frac{e^{imZ}}{1 + Z^2} = \frac{1}{Z - i} \int \frac{e^{imZ}}{Z + i} \, dZ = \frac{1}{Z - i} \varphi(z)$$

$$= \frac{1}{Z - i} \int \varphi(i) + (Z - i) \varphi'(i) + \ldots \, dZ$$

and the residue is $\varphi(i) = \frac{e^{-m}}{2i}$, and hence the value of the integral is $\frac{\pi}{2i} e^{-m}$.

Along the axis of reals $Z = X$ and along the semi-circle $Z = R(\cos \theta + i \sin \theta)$, whence $e^{imZ} = e^{-Rm \sin \theta} \int \cos(Rm \cos \theta + i \sin(Rm \cos \theta)) \, d\theta$ consequently for the given integral along the closed path we have

$$\int_{-R}^{R} \frac{e^{imX}}{1 + X^2} \, dX$$

$$+ \int_{0}^{\pi} \frac{e^{-Rm \sin \theta} \cos(Rm \cos \theta + i \sin(Rm \cos \theta))}{1 + R^2 (\cos 2\theta + i \sin 2\theta)} \, d\theta.$$
Now let \( R \to \infty \). It is not difficult to see that the last integral approaches zero as a limit because the denominator goes to infinity faster than the numerator. And hence

\[
\int_{-\infty}^{\infty} \frac{\text{e}^{imx}}{1 + x^2} \, dx = \frac{\pi}{\text{e}^m} \quad \text{but this is}
\]

\[
\int_{-\infty}^{\infty} \sqrt{\frac{\cos mX}{1 + x^2}} + i \sqrt{\frac{\sin mX}{1 + x^2}} \, dx = \frac{\pi}{\text{e}^m}.
\]

Equating real and imaginary parts, we have

\[
\int_{-\infty}^{\infty} \frac{\cos mX}{1 + x^2} \, dx = \frac{\pi}{\text{e}^m} \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\sin mX}{1 + x^2} \, dx = 0.
\]

We can write the above integral thusly

\[
2 \int_{0}^{\infty} \frac{\cos mX}{1 + x^2} \, dx = \frac{\pi}{\text{e}^m} \quad \text{or} \quad \int_{0}^{\infty} \frac{\cos mX}{1 + x^2} \, dx = \frac{\pi/2}{\text{e}^m}.
\]

Example 4. Find \( \int_{x^2 + a^2} \cos X \, dX \), where \( a > 0 \) by considering \( \int_{0}^{\infty} \frac{\text{e}^{iz}}{Z^2 + a^2} \, dZ \) around the contour consisting of the axis of reals from \(-R\) to \(R\), where \( R \) approaches
infinity, and the semi-circle which lies above the X-axis.

This function has a pole at ia and the residue is

\[
\lim_{Z \to ia} \left( \frac{e^{iz}}{Z^2 + a^2} \right) = \frac{e^{-a}}{2ia}.
\]

Hence we can write

\[
\int_{-R}^{R} f(x)dx + \int_{0}^{\pi} f(\text{Re}^{ie}) \text{Re}^{ie} i \, d\phi = 2\pi i \, \frac{e^{-a}}{2ia} = \frac{\pi e^{-a}}{a},
\]

But

\[
\int_{0}^{\pi} f(\text{Re}^{ie}) \text{Re}^{ie} i \, d\phi = 2\pi \int_{0}^{\frac{\pi}{2}} \frac{e^{-R \sin \phi}}{R^2 - a^2} R \, d\phi.
\]

\[
A = \frac{R}{R^2 - a^2} \int_{0}^{\frac{\pi}{2}} \frac{e^{-R \sin \phi}}{R^2 - a^2} \, d\phi.
\]

Hence the integral along the semi-circle tends to zero as R goes to infinity. Therefore

\[
\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + a^2} \, dx = \frac{\pi e^{-a}}{a}.
\]

But we can write the above integral as

\[
2\int_{0}^{\infty} \frac{e^{ix}}{x^2 + a^2} \, dx = \frac{\pi e^{-a}}{a},
\]

put \(e^{ix} = \cos x + i \sin x\) equating real and imaginary parts, we
\[ \int_0^\infty \frac{\cos X \, dX}{x^2 + a^2} = \frac{\pi}{2a} e^{-a}, \quad \int_0^\infty \frac{\sin X \, dX}{x^2 + a^2} = 0. \]

Example 5. - Prove
\[ \int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^x} \, dx = \frac{\pi i}{\sin \pi a}, \]
where
\[ 0 < a < 1. \]

Integrate \( f(z) = \frac{e^{az}}{1 + z} \) around the contour consisting of the X-axis and the lines \( X = \pm R, Y = \pm \frac{\pi}{2} \). The only pole within the contour is \( \frac{\pi}{2} i \), at which the residue is \( -e^{\pi i} \). Put \( Z = X + iY \),

\[ \left| f(z) \right| \leq \frac{e^{\pi X}}{e^X - 1}, \]
so that the limit, as \( R \to \infty \), approaches zero. Therefore

\[ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^x} \, dx \leq -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{a(X + \pi i)}}{1 + e^x} \, dx = -e^{\pi i} \]

Hence
\[ \frac{1 - e^{2a\pi i}}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^x} \, dx = -e^{\pi i} \]
and \[ \int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^x} \, dx = \frac{1}{\sin a \pi} \]
BIBLIOGRAPHY


