ABSTRACT

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On the Equivalence of Compactness and Finiteness in Topology

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A topological space $X$ is compact-finite if and only if compactness and finiteness are equivalent. The most commonly used term for such a space is cf. CF-spaces may be determined in many ways. However, to show that a space is cf, it suffices to prove that every compact set is finite or that every infinite set is not compact. Numerous examples and related theorems of cf-spaces are presented.
ON THE EQUIVALENCE OF COMPACTNESS AND FINITENESS IN TOPOLOGY

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I wish to acknowledge my indebtedness to my advisor, Mr. John Boyd, Jr. whose helpful suggestions and constructive criticisms made this thesis possible. My treatment of it is particularly indebted to the published work of Norman Levine[4]. Finally, my thanks to Mrs. Adelyne M. Conley who typed the final copy.

E. L. W. R.
INTRODUCTION

Topology is one of the major branches of modern mathematics. Besides the importance of this subject in its own right, an understanding of it helps immeasurably in courses in advanced calculus, real variables, complex variables, and courses in analysis. In the study of topology the undefined objects are usually called open sets. The relationship between these objects is given by a set of axioms.

This thesis is an outgrowth of an article by Norman Levine [4]. Topics covered include some topological preliminaries, an introduction to cf-space, cf-product spaces, and additivity theorems.

Each chapter begins with clear statements of pertinent definitions, principles and theorems together with illustrative examples and other descriptive materials. Numerous proofs of theorems are included.
CHAPTER I

SOME TOPOLOGICAL PRELIMINARIES

In this chapter we shall be concerned with some elementary topological preliminaries which are pertinent to the development of the thesis topic.

Definition 1. A topological space \((X,T)\) is a nonempty set \(X\) together with a family \(T\) of subsets of \(X\) possessing the following properties:

1) \(X \in T\) and \(\emptyset \in T\)

2) \(O_1 \in T\) and \(O_2 \in T\) imply \(O_1 \cap O_2 \in T\)

3) \(0_\alpha \in T\) implies \(\bigcup_\alpha O_\alpha \in T\)

In other words, the set itself and the empty set are members of the topology, the intersection of any two members of a topological space is also a member of the topology, and the union of any number of sets in a topological space is a member of the topology. The family \(T\) is called a topology for the set \(X\), and the members of \(T\) are called \(T\)-open sets or simply open sets.

Given any set \(X\) of points there are always two topologies that can be defined on \(X\). They are the discrete and indiscrete topologies. Every topology is contained in the discrete topology and contains the indiscrete topology. These two topologies are respectively the largest and smallest topology for a set \(X\).

Definition 2. The indiscrete topology for a set \(X\) is the topology in which the only open sets are \(X\) and \(\emptyset\).
Definition 3. The discrete topology for a set $X$ is a topology in which every subset of $X$ is an open set.

Closed Sets

The sets in a topology are called open sets. Throughout this discussion the closed sets shall be defined in terms of the open sets.

Definition 4. A closed set $A$ in a topological space $(X, T)$ is a set whose complement is open. If $T$ is the indiscrete topology the complement of $X$ and the complement of the empty set are the only closed sets. It is always true that the space and the empty set are closed as well as open. If $T$ is the discrete topology, then every subset is closed and open.

Theorem 1. In a topological space $X$, a subset $A$ of $X$ is open if and only if its complement is closed.

Proof. Let $A$ be open. Since $\sim(\sim A) = A$ and $A$ is open, then $\sim A$ is closed by the definition of a closed set. Conversely, let $\sim A$ be closed. Then $\sim(\sim A)$ is open by the definition of a closed set. But $\sim(\sim A) = A$. Therefore $A$ is open.

Theorem II. The intersection of any collection of closed sets is closed.

Proof. Let $\{C_\alpha\}$ be a collection of closed sets. Must show that $\bigcap_\alpha C_\alpha$ is closed. Since the collection $\{C_\alpha\}$ is closed, then $\{\sim C_\alpha\}$ is a collection of open sets by theorem I. Now, the open sets are members of $T$ and the union of open sets is a member of $T$. So, $\bigcup_\alpha \sim C_\alpha$ is open. But $\bigcup_\alpha \sim C_\alpha = \sim (\bigcap_\alpha C_\alpha)$, by De Morgan's Law. Hence $\sim (\bigcap_\alpha C_\alpha)$ is
open. Therefore, \( \bigcap \alpha C_\alpha \) is closed by the definition of a closed set.

**Functions**

The notion of a function must now be defined. The word mapping is often used as a synonym for a function.

**Definition 5.** A function \( f \) from a set \( X \) to a set \( Y \) is a rule which assigns to each \( x \in X \) a unique \( f(x) \in Y \). The set \( X \) is called the domain of \( f \). The range of \( f \) denoted by \( f(X) \) is the set of images. That is \( f(X) = \{ f(x) \mid x \in X \} \). We often express the fact that \( f \) is a function on \( X \) into \( Y \) by writing \( f: X \to Y \).

**Definition 6.** A function \( f: X \to Y \) is called one-to-one if \( f(x_1) = f(x_2) \) only when \( x_1 = x_2 \).

**Definition 7.** The image under \( f \) of \( A \) is the set of elements \( y \) in \( Y \) such that \( y = f(x) \) for some \( x \in A \subset X \).

**Definition 8.** The inverse image, denoted by \( f^{-1} \), of \( B \) is the set of those \( x \in X \) such that \( f(x) \in B \subset Y \).

**Definition 9.** A function \( f: X \to Y \) is onto if for every \( y \in Y \) there exists \( x \in X \) such that \( f(x) = y \). If \( f \) maps \( X \) onto \( Y \) and \( \{ A_\lambda \} \) is a collection of subsets of \( X \) then:

**Lemma I.** \( f \left[ \bigcup_i A_i \right] = \bigcup_i f \left[ A_i \right] \)

For inverse images, if \( \{ B_\lambda \} \) is a collection of subsets of \( Y \) then:

**Lemma II.** \( f^{-1} \left[ \bigcup_i B_i \right] = \bigcup_i f^{-1} \left[ B_i \right] \)
For $A \subseteq X$ and $B \subseteq Y$ then:

**Lemma III.** $f \left[ f^{-1} \left[ B \right] \right] \subseteq B$ and

**Lemma IV.** $A \subseteq f^{-1} \left[ f \left[ A \right] \right]$

**Definition 10.** A function $f: X \rightarrow Y$ is called an open function if the image of every open set is open. That is, if $A$ is open in $X$ then $f \left[ A \right]$ is open in $Y$.

**Definition 11.** A function $f: X \rightarrow Y$ is continuous if and only if the inverse image of every open set is open. That is if $0$ is open in $Y$, then $f^{-1} \left[ 0 \right]$ is open in $X$.

**Compact Sets**

This section is devoted to the characterization of compactness in sets.

**Definition 12.** A collection of sets $\left\{ C_{\alpha} \right\}$ is said to cover a set $A$, if and only if $A \subseteq \bigcup_{\alpha} C_{\alpha}$. If, in addition, each of the sets $C_{\alpha}$ is open, then the collection $\left\{ C_{\alpha} \right\}$ is known as an open cover. The collection $\left\{ C_{\alpha} \right\}$ is said to be finite if and only if it contains a finite number of elements.

**Definition 13.** If $\left\{ D_{\alpha} \right\}$ is a subcollection of $\left\{ C_{\alpha} \right\}$, then $\left\{ D_{\alpha} \right\}$ is a subcover of $A$ if and only if $A \subseteq \bigcup_{\alpha} D_{\alpha}$.

**Definition 14.** A set $A$ is said to be compact if and only if for every open cover $\left\{ U_{\alpha} \right\}$ of $A$, there exists a finite collection $\left\{ U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n} \right\}$, such that $A \subseteq \bigcup_{\alpha} U_{\alpha_n}$.
Theorem III. The continuous image of a compact set is compact:

Proof. Let \( f \) be a continuous function from \( X \) onto \( Y \) and \( A \) a compact subset of \( X \). Must show that \( f[A] \) is a compact subset of \( Y \). Choose \( C = \{ C_i \} \) as an open cover of \( f[A] \). By the definition of a cover \( f[A] \subseteq \bigcup_i C_i \). Now, by Lemma IV we get \( A \subseteq f^{-1}\left[ f[A] \right] \).

Hence \( A \subseteq f^{-1}\left[ \bigcup_i C_i \right] \). But, \( f^{-1}\left[ \bigcup_i C_i \right] = \bigcup_i f^{-1}[C_i] \), by Lemma II. Therefore \( A \subseteq \bigcup_i f^{-1}[C_i] \). Let \( D = \{ f^{-1}[C_i] \} \). \( D \) covers \( A \) since \( A \subseteq \bigcup_i f^{-1}[C_i] \). Now each \( C_i \) is open because the set \( C \) is open. Hence each \( f^{-1}[C_i] \) is also open since \( f \) is continuous. Therefore, \( D \) is an open cover for \( A \). But \( A \) is compact. So \( D \) must have a finite subcover, say \( f^{-1}[C_{i_1}], \ldots, f^{-1}[C_{i_m}] \) and

\[
A \subseteq f^{-1}\left[ C_{i_1} \right] \cup \ldots \cup f^{-1}\left[ C_{i_m} \right]
\]

So \( f[A] \subseteq f\left[ f^{-1}\left[ C_{i_1} \right] \cup \ldots \cup f^{-1}\left[ C_{i_m} \right] \right] \).

But \( f\left[ f^{-1}\left[ C_{i_1} \right] \cup \ldots \cup f^{-1}\left[ C_{i_m} \right] \right] = f(f^{-1}[C_{i_1}] \cup \ldots \cup f^{-1}[C_{i_m}]) \subseteq C_{i_1} \cup \ldots \cup C_{i_m} \), by Lemmata II and III. Hence the open cover \( C \) has a finite subcover. Therefore, \( f[A] \) is compact.

Theorem IV. A closed subset of a compact space is also compact.

Proof. Let \( F \) be a closed subset of a compact space \( X \) and let \( C = \{ C_i \} \) be an open cover of \( F \). By the definition of a cover \( F \subseteq \bigcup_i C_i \). But \( X = F \cup \sim F \). Hence \( X = (\bigcup_i C_i) \cup \sim F \). So \( \{ C_i \} \cup \{ \sim F \} \) is a cover for \( X \). Let \( D = \{ C_i \} \cup \{ \sim F \} \). \( D \) is an open cover for \( X \) because it is made up of open sets. But \( X \) is compact. So \( D \) must have a finite subcover, and this subcover less the set \( \sim F \) will be a finite subcover of \( F \). Hence \( F \) is compact.
Theorem V. Every closed and bounded interval on the real line $\mathbb{R}$ is compact.


Theorem VI. If $X$ has the indiscrete topology and $A \subset X$, then $A$ is compact.

Proof. $X$ has the indiscrete topology. Therefore the only open sets are $X$ and $\emptyset$. Since $X$ is the only non-empty open set every cover of $A$ is finite consisting of at most the two sets $X$ and $\emptyset$. Hence the given cover is a finite subcover of itself. So $A$ is compact.

Some Set Relations

This section is devoted to some of the notions from set theory which will be useful later.

Theorem VII. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof. $A \cap (B \cup C) = \{ x \mid x \in A, x \in B \cup C \}$

$= \{ x \mid x \in A, x \in B \text{ or } x \in C \}$

$= \{ x \mid (x \in A, x \in B) \text{ or } (x \in A, x \in C) \}$

$= \{ x \mid x \in A \cap B \text{ or } x \in A \cap C \}$

$= (A \cap B) \cup (A \cap C)$

Therefore, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Theorem VIII. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof. $A \cup (B \cap C) = \{ x \mid x \in A \text{ or } x \in B \cap C \}$

$= \{ x \mid x \in A \text{ or } (x \in B \text{ and } x \in C) \}$
\[ = \{ x \mid x \in A \cup B \text{ and } x \in A \cup C \} \]
\[ = (A \cup B) \cap (A \cup C) \]

Therefore, \( A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \)

**Definition 15.** \( A - B \) is the set of elements which belong to \( A \) but which do not belong to \( B \). In other words, \( A - B = A \cap \sim B \).

**Theorem IX.** \( A - U = A - (A \cap U) \).

**Proof.** \( A - (A \cap U) = A \cap \sim (A \cap U) \) by definition 15
\[ = A \cap (\sim A \cup \sim U) \] by De Morgan's Law
\[ = (A \cap \sim A) \cup (A \cap \sim U) \] by Theorem VII
\[ = \emptyset \cup (A - U) \] by definition 15
\[ = A - U \]

**Theorem X.** If \( A \) is closed and \( B \) is open, then \( A - B \) is closed.

**Proof.** \( A - B = A \cap \sim B \). \( \sim B \) is closed since \( B \) is open. Now \( A \) is closed and \( \sim B \) is closed. Hence by theorem II, we get that \( A \cap \sim B \) is closed. But, \( A \cap \sim B = A - B \). So, \( A - B \) is closed.

**Theorem XI.** If \( A \subseteq U \cup V \), then \( A - (A - U) \cup (A - V) = A \cap U \cap V \).

**Proof.** See sketch below.

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- \( A - V \) is denoted by (1)
- \( A - U \) is denoted by (3)
- \( A \cup V \) is denoted by (2)
CHAPTER II

AN INTRODUCTION TO CF-SPACES

In this chapter we shall discuss some properties of cf-spaces.

Definition 16. A topological space is a cf-space (compact-finite) if and only if compactness and finiteness are equivalent in $X$.

To show that a space is cf, it suffices to prove that every compact set is finite, or that every infinite set is not compact, as the following theorem shows.

Theorem XII. A finite set $A$ is compact.

Proof. Let $A$ be finite such that $A = \{x_1, x_2, \ldots, x_n\}$. Suppose $C$ is an open cover for $A$. Since $C$ is a cover for $A$, then $A \subseteq \bigcup C$. Choose $D = \{K_1, K_2, \ldots, K_n\}$ such that $x_i \in K_i$ and $K_i \subseteq C$ for each $i$. Thus $D \subseteq C$. Clearly $A \subseteq \bigcup D$ so $D$ is a subcover. So, the open cover $C$ has a finite subcover $D$. Therefore $A$ is compact.

Theorem XIII. $X$ is a cf-space if and only if each compact subset $A$ of $X$ is finite.

Proof. Assume that $X$ is a cf-space. By the definition of a cf-space compactness and finiteness are equivalent in $X$. Therefore, a subset $A$ of $X$ is compact if and only if $A$ is finite. Hence each compact subset $A$ of $X$ is finite. Conversely, assume that each compact subset $A$ of $X$ is finite. \------------ It must be shown that $X$ is cf. Let $A$ be any
compact subset of \( X \). By hypothesis \( A \) is finite. By theorem XII if \( A \) is finite then \( A \) is compact. Hence compactness and finiteness are equivalent. Therefore \( X \) is cf.

**Theorem XIV.** If \( X \) is finite, then \( X \) is a cf-space.

**Proof.** Let \( A \) be a compact subset of \( X \). Since \( X \) is finite and \( A \subseteq X \), then \( A \) must be finite. So, by theorem XIII we get that \( X \) is a cf-space.

A few examples of cf-spaces are listed below:

**Example 2-1.** A discrete topological space \( X \) is a cf-space.

**Proof.** Let \( A \) be compact and \( C = \{ \{ x \} \mid x \in A \} \) be an open cover for \( A \). Now \( C \) consists of open sets because all subsets in a discrete space are open. Furthermore, if any member of \( C \) is committed then \( C \) will not cover \( A \). So, \( C \) is the only cover of \( A \). But \( A \) is compact and by definition \( C \) must have a finite subcover. Since \( C \) is the only subcover then \( C \) is finite. Thus \( A \) is finite. So by theorem XIII we get that \( X \) is a cf-space.

**Example 2-2.** Let \( X \) be the set of positive integers and \( T = \{ \emptyset, \{1\}, \{1, 2\}, \ldots, \{x\} \} \), then \( X \) is a cf-space.

**Proof.** Suppose \( A \) is a compact subset of \( X \). Choose \( C = \{ \{1\}, \{1, 2\}, \ldots, \{1, 2, \ldots m\} \} \) as a cover for \( A \). Now \( C \) is an open cover since each member of \( C \) is an element of \( T \). Choose \( D \) as a finite subcover of \( A \). Then there is a biggest set in \( D \) say \( \{1, 2, 3, \ldots m\} \). Since all the other members of \( D \) are subsets of this set then the \( \bigcup D = \{1, 2, 3, \ldots m\} \). Now \( D \) is a subcover of \( A \), thus \( A \subseteq \bigcup D \). Hence
A \subseteq \{1, 2, \ldots, m\}$. Thus $A$ has at most $m$ elements. Therefore $A$ is finite, and by theorem XIII we get that $X$ is a cf-space.

Example 2-3. Let $X$ be any set and let $T$ consist of all subsets of $X$ which are empty or have a countable complement.

Proof: Let $A$ be a compact subset of $X$, and suppose $A$ is infinite there exists a countably infinite subset $\{a_1, a_2, a_3, \ldots\}$ of $A$. Choose $B = X - \{a_1, a_2, \ldots\}$. Now the set $\{B \cup \{a_1\}, B \cup \{a_2\}, B \cup \{a_3\}, \ldots\}$ is an open cover of $A$. For each $B \cup \{a_i\}$ is open since it is countable. For example, the complement of $B \cup \{a_i\}$ is $\{a_1, a_2, \ldots\}$. Note that all of these are needed to have a cover. For if one of the $B \cup \{a_i\}$ were omitted, then that $a_i$ is not in any other member of the cover. Therefore there is no finite subcover. But this contradicts the compactness of $A$. Hence there are no infinite compact subsets of $X$. So $X$ is a cf-space.

Example 2-4. Let $X$ be any set and let $x^*$ be a fixed point in $X$. Let $T$ consist of all subsets of $X$ which are empty or contain $x^*$. Then $(X, T)$ is a cf-space.

Proof. Let $A$ be a compact subset of $X$. By definition of $T$ the set $\{a, x^*\}$ is contained in $T$ for each $a \in A$. Choose $\{\{a, x^*\} : a \in A\}$ as an open cover of $A$. Since $A$ is compact, then $\exists$ a finite subcover of $A$. Say $\{a_1, x^*\}, \ldots, \{a_n, x^*\}$. The union of these sets has only $a_1, a_2, \ldots, a_n, x^*$ as elements which is a finite number precisely $n + 1$. Now $A \subseteq \{a_1, \ldots, a_n, x^*\}$. Therefore $A$ is finite. Hence a compact subset $A$ of $X$ is finite. Thus $X$ is cf.
CF is not invariant under continuous, 1 - 1, onto maps as shown by this example:

**Example 2-5.** Let $X$ be the set of positive integers with the discrete topology and let $Y$ consist of $0, 1, \frac{1}{2}, \ldots \frac{1}{n}, \ldots$ with the relative topology. Let $f(1) = 0, f(2) = 1, f(3) = \frac{1}{2}, \ldots, f(n) = \frac{1}{n-1}$ for $n > 1$.

**Proof.** It must be shown that $cf$ is not invariant. Since $X$ is the set of positive integers with the discrete topology, then $X$ is a $cf$-space by example 2-1. Now let $A$ be open in $Y$. Thus $f^{-1}[A] \subseteq X$. But $X$ has the discrete topology so every subset of $X$ is open. Thus $f^{-1}[A]$ is open in $X$. Therefore $f$ is continuous. Note that $f$ is so defined that no value of the range is used more than once. Thus $x \neq x_2$ implies $f(x_1) \neq f(x_2)$. So $f$ is 1-1. Similarly $f$ is onto since each value of the range is used. That is for each $y \in Y$ there exists $x \in X$ such that $f(x) = y$. Now $Y$ is compact by theorem V, but is not finite. Therefore, $X$ is not a $cf$-space. Hence $cf$ is not invariant under continuous, 1-1, onto maps.

Also $cf$ is not invariant under continuous, open, onto maps as shown by this example.

**Example 2-6.** Let $X$ be the unit interval and let $T$ consist of all subsets of $X$ which are empty or have countable complements. Let $Y$ be the set of positive integers with the indiscrete topology. Let $f(x) = 1$, if $x = 0$ or $x \in (\frac{1}{2}, 1]$, $f(x) = 2$ if $x \in (\frac{1}{3}, \frac{1}{2}]$, ..., $f(x) = n$ if $x \in (\frac{1}{n+1}, \frac{1}{n}]$, ...

**Proof.** Since $X$ is the unit interval with a topology consisting subsets
of X which are empty or have countable complements, then X is a cf-space by example 2-3. Now Y has the indiscrete topology. Thus the only open sets are Y and ∅. Thus \( f^{-1}[Y] = X \) and \( f^{-1}[∅] = ∅ \). Therefore \( f \) is continuous. The function is so defined such that each value of the range is used for some \( x \). That is for each \( y \in Y \) there exists \( x \in X \) such that \( f(x) = y \). So \( f \) is onto. Consider the interval of the form \( (\frac{1}{n}, \frac{1}{n+1}) \). Each interval of this form contains an uncountable number of points. Now let \( O \) be a non-empty open subset of X. Since X is the unit interval, then \( O \) contains all points of \([0, 1]\) except for a countable number. Say \( \{a_1, a_2, \ldots\} \). Even if \( \{a_1, a_2, a_3, \ldots\} \subset (\frac{1}{n+1}, \frac{1}{n}) \), the interval \((\frac{1}{n+1}, \frac{1}{n})\) has other points since it is uncountable. These other points are in \( O \) because \( O = [0, 1] - \{a_1, a_2, a_3, \ldots\} \). Hence \( 0 \cap (\frac{1}{n+1}, \frac{1}{n}) \neq ∅ \). For each \( n = 1, 2, 3, \ldots \), choose an \( x_n \in 0 \cap (\frac{1}{n+1}, \frac{1}{n}) \). Since \( x_n \in 0 \), then \( f(x_n) \in f(O) \) for \( n = 1, 2, \ldots \). But \( f(x_n) = n \) by the definition of the function. Therefore, \( n \in f(O) \) for \( n = 1, 2, \ldots \). Thus \( Y \subset f(O) \) because \( Y \) is the set of positive integers. But \( f(O) \subset Y \). So, \( Y = f(O) \). But \( Y \) has the indiscrete topology. So \( Y \) is open. Therefore, \( f(O) \) is open. Hence \( f \) is open since the image of an arbitrary open set in \( X \) is open in \( Y \). \( Y \) is compact by theorem VI, but clearly \( Y \) is not finite. Therefore \( Y \) is not cf. Thus cf is not invariant under continuous, onto, open maps.

CF however is invariant under open, 1-1, onto transformations as shown by the following theorem.

**Theorem XV.** Let \( X \) and \( Y \) be topological spaces and let \( X \) be cf. If \( f: X \rightarrow Y \) is open, 1-1 and onto, then \( Y \) is cf.
Proof. Since \( f: X \rightarrow Y \) is 1-1, then \( f^{-1} \) is the function defined as
\[
f^{-1}: Y \rightarrow X, \text{ and } f^{-1}(y) = X \text{ if and only if } f(x) = y.
\]
Now let \( O \) be open in \( X \). It must be shown the \( (f^{-1})^o \left[ 0 \right] \) is open in \( Y \). But \( (f^{-1})^o = f \), and \( f \left[ 0 \right] \) is open in \( Y \), since \( f \) is open. Therefore \( f^{-1} \) is continuous. Now let \( A \) be a compact subset of \( Y \). Since \( f^{-1} \) is continuous, by theorem III we get that \( f^{-1} \left[ A \right] \) is compact. That is the continuous image of a compact set is compact. Furthermore \( X \) is cf and \( f^{-1} \left[ A \right] \) is compact in \( X \). Hence \( f^{-1} \left[ A \right] \) is finite by theorem XIII. By definition \( f^{-1} \left[ A \right] = \{ x \in X \mid f(x) = A \} \), and for any set \( f \left[ f^{-1} \left[ A \right] \right] \subset A \) by Lemma III. Since \( f^{-1} \left[ A \right] \) is finite then \( f \left[ f^{-1} \left[ A \right] \right] \) is also finite because the image of any finite set is finite. But \( f^{-1} \left[ A \right] \) is finite because the image of any finite set is finite. But \( f^{-1} \left[ A \right] = A \) since \( f \) is onto. So \( A \) is finite. Thus, we have that a compact subset of \( Y \) must also be finite. Therefore \( Y \) is a cf-space. Hence cf is invariant under open, 1-1, continuous, and onto transformations.
CHAPTER III

PRODUCT SPACES WHICH ARE CF

This section is devoted to product spaces which are cf. The product space is the set product together with the Tychonoff topology described below.

**Definition 17.** The product of \((X, T)\) and \((Y, S)\) denoted by \((X, T) \times (Y, T)\), is equal to \(X \times Y = \{ (x, y) \mid x \in X \text{ and } y \in Y \} \).

**Definition 18.** A subset \(A\) of \(X \times Y\) is open if for each point \((a, b) \in A\) there exist open sets \(U \subseteq X\) and \(V \subseteq Y\) such that \(a \in U\), \(b \in V\) and \(U \times V \subseteq A\).

If there are countably many \(X_n\), then the product topology is denoted by \(\prod X_n = \{ (x_1, x_2, \ldots) \mid x_i \in X_i \}\). A set \(A\) is open if and only if for each \((x_1, x_2, \ldots) \in A\), there exist open sets \(U_1, U_2, \ldots\) such that:

1) \(U_i\) is open in \(X_i\) for all \(i = 1, 2, 3, \ldots\)
2) \(x_i \in U_i\) for all \(i = 1, 2, 3, \ldots\)
3) \(U_i = X_i\) for all but a finite number of the \(i\)'s
4) \(\bigcap U_i \subseteq A\)

We define the projection \(P_X\) as the mapping of the product \(X \times Y\) onto \(X\) and \(P_X(a, b) = a\). Likewise, the projection \(P_Y\) is the mapping of the product \(X \times Y \to Y\) where \(P_Y(a, b) = b\). These projections are continuous.
Theorem XVI. The product of compact sets is compact.

Proof. See H. L. Royden [6], p. 166.

Theorem XVII. Let $X$ and $Y$ each be cf-spaces. Then $X \times Y$ is cf.

Proof. Let $A$ be a compact subset of $X \times Y$. It must be shown that $A$ is finite. Now $P_x$ and $P_y$ are continuous, and $A$ is compact. Hence $P_x[A]$ and $P_y[A]$ are compact in $X$ and $Y$ respectively, since the continuous image of a compact set is compact. Since $A \subset X \times Y$ and $P_x[A]$ is in $X$ and $P_y[A]$ is in $Y$, then $A \subset P_x[A] \times P_y[A]$. If $(a, b) \in A$, then $P_x(a, b) \in P_x[A]$. But $P_x(a, b) = a$. So $a \in P_x[A]$. Similarly, $b \in P_y[A]$. Hence $(a, b) \in P_x[A] \times P_y[A]$. Therefore $A \subset P_x[A] \times P_y[A]$. Now $P_x[A]$ and $P_y[A]$ are finite since they are compact subsets of the cf spaces $X$ and $Y$. Thus, $P_x[A] \times P_y[A]$ is finite. Therefore, $A$ is finite since it is a subset of a finite set. So, an arbitrary compact subset $A$ of $X \times Y$ is finite. Therefore $X \times Y$ is cf.

Theorem XVII cannot be extended to the infinite case as shown by this example:

Example 3-1. For each positive integer $n$, let $X_n$ be a two point discrete topological space. Then $\prod X_n$ is compact and infinite and thus is not a cf space.

Proof. Let $X_n = \{1_n, 2_n\}$ with the discrete topology. Each $X_n$ is cf by example 2-1. Each $X_n$ is finite and hence compact. $\prod X_n$ is compact by theorem XVI. That is the product of compact sets is compact.
If $\prod X_\alpha$ is cf then each compact subset is finite. $\prod X_\alpha \subseteq \prod X_\alpha$, and $\prod X_\alpha$ is also compact. But $\prod X_\alpha$ is not finite. Therefore $\prod X_\alpha$ is not cf.

The converse of theorem XVII is true as implied by the following theorem.

**Theorem XVIII.** Let $\left\{ X_\alpha \mid \alpha \in \Delta \right\}$ be a nonempty family of nonempty spaces. If $\prod X_\alpha$ is cf, then $X_\alpha$ is cf for each $\alpha \in \Delta$.

**Proof.** Let $\Delta$ be an index set such that for each $\alpha \in \Delta$ there is an $X_\alpha$. A point taken from $\prod_{\alpha \in \Delta} X_\alpha$ is written as $(x_\alpha)_{\alpha \in \Delta}$ where $x_\alpha$ is the point chosen from $X_\alpha$. The projection $P_\alpha : \prod_{\alpha \in \Delta} X_\alpha \rightarrow X_\alpha$ is defined as $P_\alpha \left( (x_\alpha)_{\alpha \in \Delta} \right) = x_\alpha$. Now let $A$ be a compact subset of $X_\alpha$, and select $x_\alpha \in X_\alpha$ for each $\alpha \neq \alpha'$. Choose $Y_{\alpha'} = A$. Let $Y_\alpha = \left\{ x_\alpha \right\}$ for all $\alpha \neq \alpha$. Thus $Y_\alpha \subseteq X_\alpha$ for all $\alpha$. $Y_{\alpha'}$ is compact because $A$ is compact, and $Y_\alpha$ is compact because it is finite. Therefore $\prod_{\alpha \in \Delta} Y_\alpha$ is compact since the product of compact sets is compact. Since $Y_\alpha \subseteq X_\alpha$, then $\prod_{\alpha \in \Delta} Y_\alpha \subseteq \prod_{\alpha \in \Delta} X_\alpha$. By hypothesis $\prod_{\alpha \in \Delta} X_\alpha$ is cf. Hence $\prod_{\alpha \in \Delta} Y_\alpha$ is finite since it is a compact subset of a cf space. But the number of points in $\prod_{\alpha \in \Delta} Y_\alpha$ equals the numbers of points in $A$. So $A$ is finite. Hence $X_{\alpha'}$ is cf, since an arbitrary compact subset must be finite.
CHAPTER IV

ADDITIVITY THEOREMS

This last section is devoted to the relationship of the union of open and closed sets to cf-spaces.

Theorem XIX. Let X be a topological space and let $F_1$ and $F_2$ be closed subsets which are each cf. If $X = F_1 \cup F_2$, then $X$ is cf.

Proof. Let $A$ be a compact subset of $X$. By hypothesis $F_1$ is a closed set. Thus $A \cap F_1$ is closed relative to $A$. But $A$ is compact. Therefore, $A \cap F_1$ is compact by theorem IV. That is a closed subset of a compact set is compact. Furthermore $A \cap F_1 \subseteq F_1$ and $F_1$ is cf, so by definition XIII we get that $A \cap F_1$ is finite. Each compact subset of a cf-space is finite. Similarly $A \cap F_2$ is finite. Now, $A \cap F_1$ is finite, and $A \cap F_2$ is finite. Thus $(A \cap F_1) \cup (A \cap F_2)$ is finite.

By theorem VII, $(A \cap F_1) \cup (A \cap F_2) = A \cap (F_1 \cup F_2)$. But $F_1 \cup F_2 = X$. So, $(A \cap F_1) \cup (A \cap F_2) = A \cap X = A$. Therefore, $A$ is finite. Hence $X$ is a cf-space.

Theorem XIX is true for any finite number of closed sets.

Corollary I. Let $X$ be a topological space and suppose $F_1, \ldots, F_n$ are each closed in $X$ and cf. If $X = F_1 \cup \ldots \cup F_n$, then $X$ is cf.

Proof. $F_1 \cup F_2$ is cf by theorem XIX. Hence, $(F_1 \cup F_2) \cup F_3$ is cf by theorem XIX. So, $F_1 \cup F_2 \cup F_3$ is cf. Likewise $(F_1 \cup F_2 \cup F_3) \cup F_4$ is cf by theorem XIX.
So \( F_1 \cup F_2 \cup F_3 \cup F_4 \) is cf. Therefore by induction, \( F_1 \cup F_2 \cup \ldots \cup F_n \) is cf.

For open subsets of \( X \) we have the following theorem.

**Theorem XX.** Let \( X \) be a topological space and let \( U \) and \( V \) be open cf subspaces. If \( X = U \cup V \), then \( X \) is cf. To prove this theorem, the following Lemma is needed.

**Lemma I.** Let \( X \) be a cf space and suppose \( A \) is an infinite subset of \( X \). If \( a_1, \ldots, a_n \) are in \( A \), there exist open sets \( 0_1, \ldots, 0_n \) for which \( a_i \in 0_i \) and \( A - (0_1 \cup \ldots \cup 0_n) \) is infinite.

**Proof.** Suppose such open sets do not exist. This means that if we have open sets \( 0_1, \ldots, 0_n \) such that \( a_i \in 0_i \), then \( A - (\bigcup_1^n 0_i) \) is finite. Let \( \{U_\alpha\} \) be an open cover of \( A \). Therefore for each \( a_i \in A \) there exists \( U_\alpha \in \{U_\alpha\} \) such that \( a_i \in U_\alpha \). By assumption \( A - (\bigcup_1^n U_\alpha) \) is finite. Say \( A - (\bigcup_1^n U_\alpha) = b_1, \ldots, b_m \). For each \( b_i \), select \( U_\alpha \in \{U_\alpha\} \) such that \( b_i \in U_\alpha \). There is a finite number of the \( b_i \)'s. So, there is a finite number of the \( U_\alpha \)'s. Say \( U_1^*, \ldots, U_m^* \).

So we have \( U_1, \ldots, U_n \) which covers \( A - \{b_1, \ldots, b_m\} \) and \( U_1^*, \ldots, U_m^* \) which covers \( b_1, \ldots, b_m \). Thus, \( U_1, \ldots, U_n, U_1^*, \ldots, U_m^* \) covers \( A \). This cover is finite since it has \( n + m \) elements. So the open cover \( \{U_\alpha\} \) has a finite subcover. Therefore \( A \) is compact. It is given that \( A \) is infinite. So this contradicts theorem XIII. In a cf-space every compact set is finite. Therefore there exist \( 0_1, \ldots, 0_n \) as required.
With Lemma I, we proceed to prove the theorem.

Proof. Suppose $X$ is not cf. This means that there exists an infinite compact subset $A$ of $X$. Since $U$ is open by hypothesis, we have that $A \cap U$ is open relative to $A$. Of course $A$ is closed relative to $A$, so by theorem $X$ we have that $A - (A \cap U)$ is closed relative to $A$. But $A - (A \cap U)$ is merely $A - U$ so $A - U$ is closed relative to $A$. But $A - U \subseteq A$ and $A$ is compact so $A - U$ is compact by theorem IV. Also we have that $A - U \subseteq V$ since $U \cup V = X$.

By hypothesis we have that $V$ is a cf-space, so $A - U$ is a compact subset of the cf-space $V$. By definition we get that $A - U$ is finite. Similarly we obtain that $A - V$ is finite. Let $A - U = \{a_1, \ldots, a_m\}$ and $A - V = \{a_{m+1}, \ldots, a_n\}$. $A$ is the union of the three disjoint sets $A - U$, $A - V$, and $A \cap U \cap V$. Now $(A \cap U \cap V) \cup (A - U)$ is a subset of the cf-space $V$. Also we have that $(A \cap U \cap V) \cup (A - U)$ is infinite since $A$ is infinite and this set is merely $A - \{a_{m+1}, \ldots, a_n\}$. Also $\{a_1, \ldots, a_m\}$ is a finite subset of $(A \cap U \cap V) \cup (A - U)$. By lemma I we get that there exist sets $0_1, \ldots, 0_m$ such that each $0_i$ is open in $V$, $a_i \in 0_i$ for all $i = 1, \ldots, m$ and $((A \cap U \cap V) \cup (A - U)) - (0_1 \cup \ldots \cup 0_m)$ is infinite. Since $V$ is itself open by hypothesis and each $0_i$ is open in $V$ then we have that each $0_i$ is open in $X$. Also since $((A \cap U \cap V) \cup (A - U)) - (0_1 \cup \ldots \cup 0_m)$ is infinite then $((A - V) \cup (A \cap U \cap V) \cup (A - U)) - (0_1 \cup \ldots \cup 0_m)$ is infinite. In other words $A - (0_1 \cup \ldots \cup 0_m)$ is infinite. Now we have that $A - (0_1 \cup \ldots \cup 0_m)$ is an infinite subset of the cf-space $U$. Also $a_{m+1}, \ldots, a_n$ are in $A - (0_1 \cup \ldots \cup 0_m)$ so by lemma I, there exist
sets \(0_{m+1}, \ldots, 0_n\) such that each is open in \(U\), \(a_i \in 0_i\) for all 
\(i = m + 1, \ldots, n\) and \((A - (0_i \cup \ldots \cup 0_m)) - (0_{m+1} \cup \ldots \cup 0_n)\) 
is infinite. From this we get that \(A - (0_i \cup \ldots \cup 0_n)\) is infinite 
by the fact from set theory which says \((A - B) - C = A - (B \cup C)\). 
Since \(U\) is open \(X\) then \(0_{m+1}, \ldots, 0_n\) are also open in \(X\). Since 
\((A - U) \cup (A - V) = \{a_1, \ldots, a_n\}\) we have that \((A - U) \cup (A - V)\) 
\(\subseteq 0_i \cup \ldots \cup 0_n\). Thus \(A - (0_i \cup \ldots \cup 0_n) \subseteq A \cap U \cap V \subseteq U \cap V\). 
But \(U \cap V\) is a cf-space because it is a subspace of the cf-space \(U\). Hence \(A - (0_i \cup \ldots \cup 0_n)\) is an infinite subset of the 
cf-space \(U\) so \(A - (0_i \cup \ldots \cup 0_n)\) is not compact. Which means 
\(A - (0_i \cup \ldots \cup 0_n)\) is not compact in \(X\). Compactness is not a 
relative property. Thus there exists an infinite open cover \(\Phi\) of 
\(A - (0_i \cup \ldots \cup 0_n)\) with no finite subcover. Hence \(\Phi\) 
\(\bigcup \{0_i, \ldots, 0_n\}\) is an open cover of \(A\) with no finite subcover. For if there were a finite subcover, then a finite number of 
sets would come from \(\Phi\) and these would have to cover 
\(A - \{0_i, \ldots, 0_n\}\), but \(\Phi\) has no finite subcover. Thus we have 
contradicted the compactness of \(A\). Therefore the assumption that \(X\) is 
not cf is false. Hence \(X\) is cf.

It seems appropriate that we present this example.

**Example 4-1.** Let \(X\) consist of the positive integers and let \(T\) consist 
of all subsets of \(\{2, 3, \ldots, n, \ldots\}\) together with all subsets of \(X\) 
containing 1 which have finite complements. Let \(F = \{1\}\) and let 
\(G = \{2, 3, \ldots, n, \ldots\}\). Then \(G\) is open, \(F\) is closed, \(F\) and \(G\) are 
each cf, \(X = F \cup G\), but \(X\) is not cf.
Proof. G is open by the definition of T. F is the complement of G. Therefore F is closed. Furthermore, F is finite. Thus by theorem XIV F is cf. Note that all subsets of G are open. So G has the discrete topology. So by example 2-1, G is cf. Clearly X = F \cup G since X consists of the positive integers. We will show that X is compact. Let \( \{0_\alpha\} \) be an open cover of X. Then there exists \( \alpha^* \) such that \( 1 \in 0_{\alpha^*} \). Thus the complement of 0_{\alpha^*} is finite, say \( \{a_1, \ldots, a_n\} \). Choose \( 0_{a_1}, \ldots, 0_{a_n} \supset 0_i \in 0_\alpha \). Hence \( \{0_{\alpha^*}, 0_{a_1}, \ldots, 0_{a_n}\} \) covers X and is a finite subcover of the open cover \( \{0_\alpha\} \). Therefore X is compact. Now X is compact but not finite. So X is not a cf-space.
This appendix is devoted to the special symbols used in this thesis.

\(\in\) \quad \text{belongs to}

\(\subset\) \quad \text{is contained in}

\(=\) \quad \text{equals}

\(\emptyset\) \quad \text{the empty set}

\(A\) \quad \text{a collection of sets}

\(\forall\) \quad \text{for all}

\(\sim A\) \quad \text{the complement of } A

\(\exists\) \quad \text{there exist}

\(f : X \rightarrow Y\) \quad \text{the mapping of } X \text{ into } Y

\(f[A]\) \quad \text{the image of } A \text{ under the mapping } f

\(A - B\) \quad \text{points of } A \text{ not in } B

\(f^{-1}[B]\) \quad \text{inverse of } B \text{ under the mapping } f

\(\cap\) \quad \text{intersection}

\(\cup\) \quad \text{union}

\(X \times Y\) \quad \text{the product of } X \text{ and } Y

\((x, y)\) \quad \text{an ordered pair of } x \text{ and } y

\(\exists\) \quad \text{such that}

\([a, b]\) \quad \text{half open-interval}

\((X, T)\) \quad \text{a topological space}

\(T\) \quad \text{a topology}

\(X\) \quad \text{a set}

\(\prod x_n\) \quad \text{the product of } x \text{ where } n \text{ is infinite.}

\(\Lambda\) \quad \text{an index set}
BIBLIOGRAPHY


